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MATHEMATICS OF MODERN ENGINEERING

VOLUME II

(Mathematical Engineering)

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PREFACE

The two purposes of this book are implied in its title and subtitle. Its first purpose (along with that of the first volume and a third, which is in preparation) is to present those aspects of mathematics which the experience of a large manufacturing organization, in dealing with electrical and mechanical investigations, has found to be of most value to engineers. The mathematical material treated is not the selection of one or two individuals of what they consider mathematically useful in the engineering work of a large engineering organization, but the composite opinion based on extensive experience of engineers and physicists who apply themselves to the abstract principles of engineering in daily engineering research and practice.

The second purpose, which is even more important, is to present an introduction to the methods of mathematical engineering by the analysis of discrete physical systems. There has developed during the last two decades a phase of engineering which may properly be called mathematical engineering. Its analogue in physics is mathematical physics. Many similarities exist in the two fields. The subject materials of mathematical physics and mathematical engineering are respectively physics and physics and engineering; the tool of both fields is mathematics, regardless of how simple or advanced the mathematics may be. Mathematical physics is not restricted to one branch of physics. Neither is mathematical engineering confined to one branch of engineering, because the fundamental method of analysis in mathematical engineering remains the same regardless of which branch of engineering is practiced. The similarities of the two subjects are pointed out in the introduction.

The material and methods of this book have evolved during the last decade out of the research engineering work and the Advanced Course in Engineering * of the General Electric Company. Although prepared for that course, the book should be just as useful in graduate engineering work at universities since the material has been tempered by use in the instruction of students not only in the course of the

* A. R. Stevenson, Jr., and Alan Howard, "An Advanced Course in Engineering," *Trans. A.I.E.E.*, March, 1935. A. R. Stevenson, Jr., and Simon Ramo, "A New Postgraduate Course in Industry in High-Frequency Engineering," *Electrical Engineering*, July, 1940.

General Electric Company but also in the courses for graduate students in more than one university. It is thus a joint product of the engineering office and the university classroom.

I am indebted to all whose publications have been used or cited, but, in particular, to nine friends it is a pleasure to express my gratitude.

Without the encouragement of Dr. A. R. Stevenson, Jr., Staff Assistant to the Vice President of Engineering of the General Electric Company, Volumes II and III would not have been written. As one of the two originators of the Advanced Course he has continually guided, directly and indirectly, the preparation of the manuscript in order that it might be adapted to the needs and methods of the course.

I am grateful to Dr. Stevenson, to Dr. Saul Dushman, Assistant Director of the Research Laboratory, and to Mr. P. L. Alger, Staff Assistant to the Vice President of Engineering, of the General Electric Company, for their generous aid in numerous projects of which this text is one.

In Chapter II much use has been made of certain papers of Mr. Gabriel Kron, consulting engineer of the company. His generosity is deeply appreciated.

Valuable suggestions regarding both form and content (for Volumes I. II, and III) have been made by Messrs. Alan Howard, B. R. Prentice, and T. C. Johnson, who, during the preparation of the manuscript, have been in succession in charge of the Advanced Course in Engineering of the General Electric Company.

I thank Mr. A. B. Chafetz for his aid in checking numerical calculations and drafting and Mr. Delbert Zilmer for his careful reading of the galley proofs.

ERNEST G. KELLER

Burbank, California
April, 1942

INTRODUCTION

Mathematical engineering consists of those parts of all branches of engineering which can be formulated mathematically.

The *fundamental method* of mathematical engineering consists of the two processes· (*a*) reduction of the physical phenomena involved to a mathematical system, (*b*) solution of the system. The two processes in this text are called, for brevity, *set-up* and *solve.* The first process requires, in addition to a knowledge of mathematical physics and engineering, originality and inventive ability in thought. The second process requires mathematical knowledge. In general, the first process is a difficult one.

Mathematical engineering naturally resolves itself into *two divisions*. The first division may be called the analysis of discrete engineering systems. It consists of those problems which involve a finite number of variables or a finite number of degrees of freedom. Frequently, these problems reduce mathematically to systems of a finite number of ordinary linear or non-linear, differential or integral equations. Examples of problems of the first division are the analyses of linear and non-linear networks, rotating electrical machines, airplane motions, locomotive oscillations, and vibrations of motors and machines. Problems in probability, statistics, and applications of number theory to machine windings also belong to this division. The second division may be entitled the analysis of continuous engineering systems. Field problems in aerodynamics, hydrodynamics, electrodynamics, and elasticity belong to this division. Frequently, such problems reduce mathematically to systems of partial differential equations.

This volume is concerned with the analyses of discrete engineering systems. An attempt has been made throughout to place equal emphasis on the two processes, set-up and solve.

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Chapter III

Non-Linearity in Engineering

MATHEMATICS OF MODERN ENGINEERING

CHAPTER I

ENGINEERING DYNAMICS AND MECHANICAL VIBRATIONS

(1) Calculus of Variations, (2) Hamilton's Principle, (3) Lagrange's Equations, (4) Lagrange's Equations and the Theory of Vibrations, (5) Lagrange's Equations and Holonomic Systems, (6) Non-holonomic Systems, (7) Energy Method and Rayleigh's Principle, (8) Additional Methods and References.

All engineering problems in Volume I were reduced to mathematical systems by means of (*a*) Newton's laws of motion, (*b*) Kirchhoff's laws of electric circuits, and (*c*) the laws of vector analysis. There exist more general principles which include the above principles as special cases.

Mathematical physicists have long sought a single principle from which all other physical principles can be drawn. The most fundamental single principle of mathematical physics is Hamilton's principle. The fundamental equations of dynamics as well as the Maxwell field equations, equations of elasticity, and other basic systems are derivable from Hamilton's principle. The fundamental equations of dynamics and Rayleigh's principle yield, as a special field, the theory of vibrations.

Hamilton's principle is most easily understood and derived in the notation of the calculus of variations. The calculus of variations itself has many applications in engineering aside from its use in establishing Hamilton's principle, but the proof of Hamilton's principle for the field of dynamics is sufficient justification for the study of this branch of mathematics.

(1)

Calculus of Variations

The calculus of variations deals with problems in maxima and minima. It is recalled from the calculus that in the elementary theory of maxima and minima the problem is to determine those values of the independent variables $(x_1, x_2, \cdots, x_n)$ for which the function $y =$

$f(x_1, x_2, \cdots, x_n)$ takes on either maximum or minimum values. In the elementary calculus of variations a definite integral, whose integrand is a function of one or more unknown functions and their derivatives, is given. The problem then is to find the unknown function (or functions) which will render the definite integral a maximum or minimum. Because the easiest problems in the calculus of variations are concerned with geometrical properties this section begins with a simple geometrical problem.

1·1. Introductory Problem. Let it be required to find, by the calculus of variations, the equation of the shortest curve joining two points P_1 and P_2. It is not necessary to consider a curve existing in three-space and joining P_1 and P_2, since the projection of such a curve onto a plane containing P_1 and P_2 is shorter than the curve itself. Thus the shortest curve is sought only among curves which lie wholly in a plane containing P_1 and P_2. Moreover, as possible shortest curves, only single-valued curves (functions) and curves which are continuous and on which the tangent turns continuously need be considered. Such curves in the calculus of variations are said to be of **class *C* prime.** Any single-valued function which is continuous and possesses a continuous first derivative is defined to be of **class *C′*.** The curves, among which the curve is sought which minimizes the given integral, are called **admissible arcs** or curves. For engineering purposes the properties so far specified for the admissible arcs may be viewed as assumptions under which the solution is sought. The assumptions may be changed. In the calculus of variations this actually happens; the curves admissible for one problem may not be admissible for another. In general, more restrictions put on the admissible arcs render the analysis easier, but the results are accordingly restricted in value.

When a function is said to be of class C' it is understood to be of class C' in the interval $x_1 \leqq x \leqq x_2$. To find an extremum (minimum or maximum) means in this chapter to find only Euler's necessary condition. No sufficiency condition is implied.

The analytical statement of the problem now is: Let it be required to find among the admissible arcs joining P_1 and P_2 that one, $y = y(x)$, which minimizes the integral

$$I = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx,$$

where x_1 and x_2 are the abscissas of P_1 and P_2, respectively, and $y' = dy/dx$. The admissible curves joining P_1 and P_2 (Fig. 1·1) may then be represented analytically by the equation

$$y = y(x) + \alpha\eta(x), \qquad [1]$$

where α is a parameter independent of x and $\eta(x)$ is an arbitrary function which vanishes at x_1 and x_2. If the value of y from Eq. (1) is substituted in the integral of the problem, there results

$$I(\alpha) = \int_{x_1}^{x_2} \{1 + [y'(x) + \alpha\eta'(x)]^2\}^{1/2}\,dx,$$

where the primes denote derivatives with respect to x. Since the limits of the integral are constants, $I(\alpha)$ is a function of the single parameter

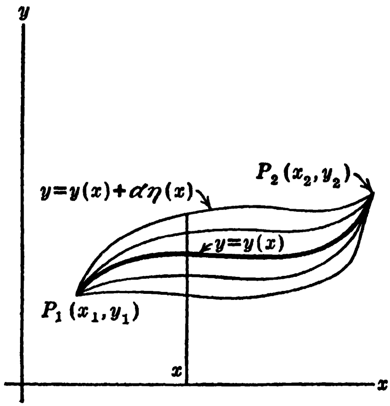

FIG. 1·1

α. If $I(\alpha)$ is to be a minimum for $\alpha = 0$, it is necessary that $dI(\alpha)/d\alpha$, denoted by $I'(\alpha)$, vanish for $\alpha = 0$. Since the limits x_1 and x_2 are constants, it follows, by the rule for differentiation of an integral with respect to a parameter, that

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{d}{d\alpha}\{1 + [y'(x) + \alpha\eta'(x)]^2\}^{1/2}\,dx$$

$$= \int_{x_1}^{x_2} \frac{y'(x) + \alpha\eta'(x)}{\{1 + [y'(x) + \alpha\eta'(x)]^2\}^{1/2}}\,\eta'(x)dx$$

and

$$I'(0) = \int_{x_1}^{x_2} \frac{y'\eta'(x)dx}{(1 + y'^2)^{1/2}}.$$

Integration by parts and use of the fact that $I'(0)$ must vanish yield

$$I'(0) = \frac{y'\eta(x)}{[1 + (y')^2]^{1/2}}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left\{\frac{y'}{[1 + (y')^2]^{1/2}}\right\}\eta(x)dx = 0.$$

By the definition of $\eta(x)$ it follows that $\eta(x_2) = \eta(x_1) = 0$. Consequently

$$I'(0) = -\int_{x_1}^{x_2} \frac{d}{dx}\left\{\frac{y'}{\sqrt{1+(y')^2}}\right\} \eta(x)dx = 0. \qquad [2]$$

It can be shown without much difficulty (see Ex. 7) that, since $\eta(x)$ is an arbitrary function, the last integral can vanish only if

$$\frac{d}{dx}\left\{\frac{y'}{\sqrt{1+(y')^2}}\right\} = 0.$$

The first integral of this equation is $\dfrac{y'}{\sqrt{1+y'^2}} = C_1$. A second integration gives

$$y = \frac{C_1}{\sqrt{1-C_1^2}}x + C_2. \qquad [3]$$

When the arbitrary constants C_1 and C_2 have been determined so that (3) passes through the points P_1 and P_2, the required minimizing curve has been found.

1·2. Euler's Equations: First Necessary Condition for Simplest General Case. The integral which was minimized in §1·1 is a special form of the more general integral

$$I = \int_{x_1}^{x_2} F(x, y, y')dx. \qquad [4]$$

Let it be required to find among the admissible arcs joining P_1P_2 (Fig. 1·1) that one, $y = y(x)$, which minimizes I. An admissible arc, in this more complicated theory, has precisely the properties[1] prescribed in §1·1. If $y = y(x)$ be the minimizing curve and Eq. (1) be substituted in (4), evidently

$$I(\alpha) = \int_{x_1}^{x_2} F[x, y + \alpha\eta(x), y' + \alpha\eta'(x)]dx. \qquad [5]$$

In order that $I(\alpha)$ take on a minimum (or maximum) value for $\alpha = 0$ it is necessary that

$$I'(0) = \int_{x_1}^{x_2} [F_y\eta(x) + F_{y'}\eta'(x)]dx = 0, \qquad [6]$$

[1] Usually the admissible arcs are taken to be curves which are continuous and consist of a finite number of arcs on each of which the tangent turns continuously, i.e., the curve may have corners. The results, however, are sufficiently general for the present purpose.

where F_y and $F_{y'}$ denote respectively the partial derivatives of $F(x, y, y')$ with respect to y and y'. If the formula for integration by parts, $\int u\,dv = uv - \int v\,du$, is applied to (6), where

$$u = F_{y'} \qquad v = \eta(x)$$

$$du = \frac{dF_{y'}}{dx} \qquad dv = \eta'(x)dx$$

there results

$$I'(0) = \int_{x_1}^{x_2} F_y\eta(x)dx + F_{y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{dF_{y'}}{dx}\,\eta(x)dx = 0.$$

Since

$$\eta(x_2) = \eta(x_1) = 0,$$

and

$$I'(0) = \int_{x_1}^{x_2}\left[F_y - \frac{d}{dx}F_{y'}\right]\eta(x)dx = 0,$$

the reasoning following Eq. (2), or Ex. 7, yields

$$F_y - \frac{d}{dx}F_{y'} = 0. \qquad [7]$$

Now $F_{y'}(x, y, y')$ is a function of the three variables x, y, y', and y and y' are in turn functions of x. By the formula for total derivative

$$\frac{dF_{y'}}{dx} = F_{y'y'}\frac{dy'}{dx} + F_{y'y}\frac{dy}{dx} + F_{y'x}.$$

Thus Eq. (7) may be written

$$F_{y'y'}y'' + F_{y'y}y' + F_{y'x} - F_y = 0, \qquad [8]$$

where

$$F_{y'y'} = \frac{\partial^2 F}{\partial y'^2}$$

and

$$F_{y'x} = \frac{\partial^2 F}{\partial y'\partial x}.$$

Equation (7), or (8), is known in the calculus of variations as **Euler's equation.** This equation is the first necessary condition which $y = y(x)$ must satisfy in order that this function may render I a minimum.

There is the theorem: Every function $y = y(x)$ which minimizes or maximizes the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')dx$$

must satisfy Eq. (7). It is recalled from the calculus that $f'(a) = 0$ is not sufficient to insure the existence of a minimum of $f(x)$ at $x = a$. It is necessary also that $f''(a) > 0$. The two conditions are both necessary and sufficient to insure a minimum. Hence in the calculus of variations, it is natural to expect that $y = y(x)$ must satisfy further conditions in order that it maximize or minimize (4). These conditions are more complicated than Euler's equation and are not essential for our purpose. In many practical problems these additional conditions may be waived, at least in a first treatment. Equation (8) is a differential equation of the second order. Its general solution contains two arbitrary constants and hence represents a two parameter family of curves. These solutions are called **extremals** and the curves of the family **extremal arcs.**

EXAMPLE. Plane curves of constant density join the points P_1 and P_2 which subtend an angle of less than 60° at the origin. Find the equation of the curve of class C' which has the least moment of inertia with respect to the origin.

Let the two points be $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ and the polar equations of the curves joining these points be $r = r(\theta)$. The integral to be minimized evidently is

$$I = \int r^2\, ds = \int_{r_1}^{r_2} r^2[1 + r^2\theta'^2]^{\frac{1}{2}}\, dr.$$

That particular $r = r(\theta)$ which minimizes I must satisfy Euler's equation

$$F_\theta - \frac{d}{dr} F_{\theta'} = 0,$$

where

$$F = r^2[1 + r^2\theta'^2]^{\frac{1}{2}},$$

and thus

$$F_\theta = 0,$$

and

$$F_{\theta'} = \frac{r^4\theta'}{[1 + r^2\theta'^2]^{\frac{1}{2}}}.$$

Euler's equation then reduces to

$$\frac{d}{dr}\left[\frac{r^4\theta'}{(1 + r^2\theta'^2)^{\frac{1}{2}}}\right] = 0.$$

The first integral of this equation is

$$\frac{r^4\theta'}{(1 + r^2\theta'^2)^{1/2}} = c,$$

or

$$r^8\theta'^2 = c^2(1 + r^2\theta'^2).$$

The last equation may be written

$$\theta = \int \frac{c\,dr}{r\sqrt{r^6 - c^2}}.$$

The integral on the right is easily evaluated by means of the substitution $r^3 = c \sec z$. The relation between θ and r then is

$$\theta = \frac{1}{3}\tan^{-1}\frac{\sqrt{r^6 - c^2}}{c} + c_1,$$

$$r^3 = c \sec 3(\theta - c_1).$$

By proper choice of c and c_1 the graph of the last equation passes through P_1 and P_2. This function of θ is a solution of Euler's equation and moreover, the last equation is the equation of the minimizing curve required.

EXERCISES AND PROBLEMS I

1. Show that the minimum surface of revolution generated by revolving about the x axis a curve of class C' joining P_1 and P_2 is a catenary. The integral to be minimized is

$$I = 2\pi \int_{x_1}^{x_2} y(1 + y'^2)^{1/2}\,dx$$

$$F = y(1 + y'^2)^{1/2}, \quad F_y = (1 + y'^2)^{1/2}, \quad F_{y'} = \frac{yy'}{(1 + y'^2)^{1/2}}$$

and

$$F_y - \frac{d}{dx}F_{y'} = 0 \quad \text{is} \quad (1 + y'^2)^{1/2} - \frac{d}{dx}\left[\frac{yy'}{(1 + y'^2)^{1/2}}\right] = 0.$$

Remembering that both y and y' are functions of x and differentiating out, there is obtained for Euler's equation

$$yy'' - (y')^2 - 1 = 0. \quad \text{Set } y' = p.$$

Then

$$y'' = p\frac{dp}{dy}$$

and the differential equation is

$$\frac{p\,dp}{p^2 + 1} = \frac{dy}{y}.$$

2. Minimize the integral $\int_{t_1}^{t_2} (m^2x'^2 + n^2x^2)dt$.

3. Show that the minimum line upon a sphere joining two points of the surface is the arc of a great circle. First show that the integral to be minimized is

$$I = \int_{\varphi_1}^{\varphi_2} R\left[1 + \cos^2\varphi\left(\frac{d\theta}{d\varphi}\right)^2\right]^{1/2} d\varphi,$$

remembering that the spherical surface coordinates R, θ, φ are related to x, y, z by the relations

$$x = R\cos\varphi\cos\theta$$

$$y = R\cos\varphi\sin\theta$$

$$z = R\sin\varphi.$$

4. Show that the minimum line upon a circular cylinder is a helix.

5. A particle of mass m falls from rest on a curve joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. It is assumed that the particle moves without friction on the curve. Find the equation of the curve for which the time of descent is a minimum. The integral to be minimized is

$$I = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\left[\frac{1 + y'^2}{y - a}\right]^{1/2} dx,$$

where $a = y_1$ and g is the acceleration of gravity.

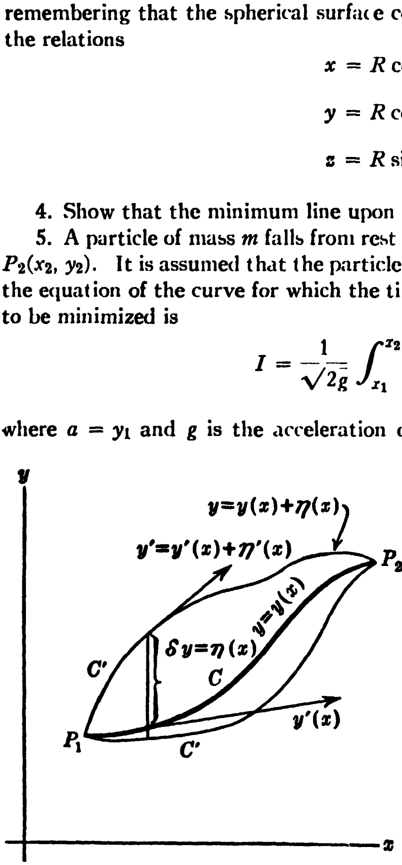

FIG. 1·2

6. Find the minimum line on a cone of revolution.

7. If M is a function of x which is continuous in the interval $x_1 \leqq x \leqq x_2$ and if

$$\int_{x_1}^{x_2} M\eta(x)dx = 0$$

for all functions η which vanish at x_1 and x_2 and which are of class C', then show that $M = 0$ in $x_1 \leqq x \leqq x_2$.

1·3. Euler's Equation by Means of Variations.

Equation (7) will now be obtained by the so-called method of variations. In Fig. 1·2 let C be the arc, whose equation is $y = y(x)$, minimizing (or maximizing) the integral (4). Let C', a neighboring curve, be given by $y = y(x) + \eta(x)$, where $\eta(x)$ is the increment in y on passing from C to C', x being fixed. Then

(a) y' becomes $y' + \eta'(x)$ on passing from C to C',

(b) $\eta(x)$ is called the **variation** of y and is denoted by δy,

(c) $\eta'(x) = \dfrac{d}{dx}\,\delta y$, and

(d) $\eta'(x) =$ change in the slope of y on passing from C to C' (x being fixed) $= \delta \dfrac{dy}{dx}$.

Thus (c) and (d) give the very important relation

$$\frac{d}{dx}\,\delta y = \delta \frac{dy}{dx} \quad \text{or} \quad d\,\delta y = \delta\,dy. \tag{9}$$

That is, the derivative of the variation is the variation of the derivative. It is remembered from the calculus that the symbol d applies to changes taking place along a particular curve. From (a), (b), (c), and (d) it is evident that δ applies to changes which occur on passing from the curve C to a curve C'. It is natural to expect that $\delta F(y)$, $\delta F(y, y')$, etc., are computed by the same formulas as $dF(y)$, $dF(y, y')$, etc. In fact the proof of the variation formulas follows the proof for the differential formulas.

It is easy to establish the formulas:

$$\begin{aligned} \delta F(y) &= \frac{\partial F}{\partial y}\,\delta y = F_y\,\delta y, \\ \delta F(y, y') &= \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y' = F_y\,\delta y + F_{y'}\,\delta y', \\ \delta F(x, y, y') &= \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y' = F_y\,\delta y + F_{y'}\,\delta y'. \end{aligned} \tag{10}$$

Now $\delta F(y)$ is defined by the equation

$$\delta F(y) \equiv F(y + \delta y) - F(y).$$

By Taylor's series $F(y + \delta y) = F(y) + \dfrac{\partial F}{\partial y}\,\delta y + \cdots$ higher powers [2] in δy. Thus

$$\begin{aligned} \delta F(y) &= F(y) + \frac{\partial F}{\partial y}\,\delta y + \cdots - F(y) \\ &= \frac{\partial F}{\partial y}\,\delta y + \cdots \text{ higher powers in } \delta y \text{ which are} \end{aligned}$$

neglected as in the case of the differential $d\,F(y)$.

[2] The neglect of the higher powers in δy restricts the nature of the admissible curves. However, the results are sufficiently general for the purpose at hand. See *Calculus of Variations* by G. A. Bliss and *Lectures on the Calculus of Variations* by Oskar Bolza.

Likewise

$$\delta F(y, y') \equiv F(y + \delta y, y' + \delta y') - F(y, y')$$
$$= F_y\, \delta y + F_{y'}\, \delta y'.$$

Also

$$\delta F(x, y, y') \equiv F(x + \delta x, y + \delta y, y' + \delta y') - F(x, y, y')$$
$$= F_x\, \delta x + F_y\, \delta y + F_{y'}\, \delta y'.$$

In $F(x, y, y')$ it is understood that y and y' are functions of x and that when the variation of $F(x, y, y')$ is taken x is held constant. Consequently $\delta x = 0$ and

$$\delta F(x, y, y') = \frac{\partial F}{\partial y}\, \delta y + \frac{\partial F}{\partial y'}\, \delta y'.$$

By comparing formulas (10) with the formulas for total differentials, it is seen that δ operates like the symbol d. Euler's equation can now be expressed in a different form. It is recalled from the calculus that

$$\frac{d}{d\alpha}\int_{x_1}^{x_2} F[x, y(\alpha), y'(\alpha)]dx = \int_{x_1}^{x_2} \frac{d}{d\alpha}\, [F(x, y(\alpha), y'(\alpha))]dx,$$

where x_1 and x_2 are constants. Likewise

$$\delta \int_{x_1}^{x_2} F(x, y, y')dx = \int_{x_1}^{x_2} \delta F(x, y, y')dx.$$

It is now easily seen that the equation

$$\delta \int_{x_1}^{x_2} F(x, y, y')dx = \int_{x_1}^{x_2} [F_y\, \delta y + F_{y'}\, \delta y']dx = 0 \qquad [11]$$

yields Euler's equation. Since

$$\int_{x_1}^{x_2} F_{y'}\, \delta y'\, dx = \int_{x_1}^{x_2} (F_{y'} \frac{d}{dx}\, \delta y)dx$$

an integration by parts applied to the last integral yields

$$\int_{x_1}^{x_2} (F_{y'} \frac{d}{dx}\, \delta y)\, dx = F_{y'}\, \delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y \frac{d}{dx}\, F_{y'}\, dx.$$

Thus Eq. (11) becomes

$$\delta \int_{x_1}^{x_2} F(x, y, y')dx = \int_{x_1}^{x_2} (F_y - \frac{d}{dx}\, F_{y'})\delta y = 0.$$

But $\delta y = \eta(x)$ is an arbitrary function of x and by the same reasoning employed following Eq. (2) it follows that

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

This is Euler's equation. Beginning with this equation and tracing backward the steps displayed, we obtain Eq. (11). Thus Eqs. (7) and (11) are equivalent, i.e., each implies the other.

If $\delta I = \int_{x_1}^{x_2} \delta F(x, y, y')dx = 0$, then the integral I is said to be **stationary.** Stationary integrals play a very important role in mathematical physics.

EXAMPLE. Let it be required to find, by the method of variations, the equation in polar coordinates of the shortest arc connecting $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$. The integral to be minimized is

$$I = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta.$$

The first necessary condition to be satisfied is $\delta I = 0$. Accordingly

$$\begin{aligned}
\delta I &= \int_{\theta_1}^{\theta_2} \delta(r^2 + r'^2)^{\frac{1}{2}} d\theta \\
&= \int_{\theta_1}^{\theta_2} \left[\frac{r\,\delta r}{(r^2 + r'^2)^{\frac{1}{2}}} + \frac{r'\,\delta r'}{(r^2 + r'^2)^{\frac{1}{2}}}\right] d\theta \\
&= \int_{\theta_1}^{\theta_2} \left[\frac{r\,\delta r}{(r^2 + r'^2)^{\frac{1}{2}}} + \frac{r' \dfrac{d}{d\theta}\delta r}{(r^2 + r'^2)^{\frac{1}{2}}}\right] d\theta.
\end{aligned}$$

In the formula $\int u\, dv = u\, v - \int v\, du$, let

$$u = \frac{r'}{(r^2 + r'^2)^{\frac{1}{2}}}, \quad dv = \frac{d}{d\theta}\delta r,$$

$$du = \left[\frac{r'}{\sqrt{r^2 + r'^2}}\right]' d\theta, \quad v = \delta r.$$

Then

$$\delta I = \int_{\theta_1}^{\theta_2} \left\{\frac{r}{(r^2 + r'^2)^{\frac{1}{2}}} - \frac{d}{d\theta}\left[\frac{r'}{\sqrt{r^2 + r'^2}}\right]\right\} \delta r\, d\theta = 0.$$

The variation δr is an arbitrary function of θ and by the usual reasoning it follows that

$$\frac{r}{\sqrt{r^2 + r'^2}} - \frac{d}{d\theta}\left[\frac{r'}{\sqrt{r^2 + r'^2}}\right] = 0,$$

or performing the indicated differentiation, there results

$$\frac{rr'' - 2r'^2 - r^2}{(r^2 + r'^2)^{3/2}} = 0.$$

The negative of the left side of the last equation is the formula for curvature in polar coordinates. The extremals of the problem, then, are arcs of zero curvature and that one which passes through P_1 and P_2 is the minimizing arc required.

EXERCISES AND PROBLEMS II

1. Obtain the required differential equations for each of the first six problems of problem set I by means of setting the first variation equal to zero.

1·4. Generalization of Simplest Case: More than One Independent Variable. Let there be given in the xy plane the curve $C: f(x, y) = 0$. Let it be required to find the surface $z = g(x, y)$, Fig. 1·3, passing through $f(x, y) = 0$ which shall minimize the integral

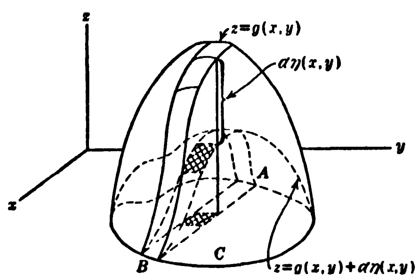

FIG. 1·3

$$I = \iint_S F(x, y, z, p, q)dx\,dy,$$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, and S is the area in the xy-plane bounded by C.

The partial differential equation defining $z = g(x, y)$ is obtainable either by substituting in I

$$z = g(x, y) + \alpha\eta(x, y)$$

and proceeding as in §1·2 or by setting $\delta I = 0$ as in §1·3. By the second method, remembering that both x and y are independent variables and employing Eqs. (10), we have

$$\delta I = \iint_S F(x, y, z, p, q)dx\,dy = \iint_S (F_z\delta z + F_p\delta p + F_q\delta q)dx\,dy = 0. \qquad [12]$$

Since by Eq. (9), $\delta p = \frac{\partial}{\partial x}\,\delta z$ and $\delta q = \frac{\partial}{\partial y}\,\delta z$, it follows that

$$\iint_S F_p\,\delta p\,dx\,dy = \iint_S \left(F_p \frac{\partial}{\partial x}\,\delta z\right)dx\,dy.$$

The last integral is converted by an integration by parts into

$$\int F_p\,\delta z \Big|_A^B dy - \iint_S \frac{d}{dx} F_p\,dx\,dy\,\delta z,$$

where A and B are points on $f(x, y) = 0$ whose ordinates are equal. Since $\delta z = 0$ at these points it follows that

$$\iint_S F_p\,\delta p\,dx\,dy = -\iint_S \frac{d}{dx} F_p\,dx\,dy\,\delta z.$$

In an identical manner

$$\iint_S F_q\,\delta q\,dx\,dy = -\iint_S \frac{d}{dy} F_q\,dx\,dy\,\delta z,$$

and thus Eq. (12) becomes

$$\delta I = \iint_S \left(F_z - \frac{d}{dx} F_p - \frac{d}{dy} F_q\right) \delta z\,dx\,dy = 0.$$

It can be shown (see Ex. 2) that, since δz is an arbitrary function of x and y, the last integral vanishes only in case

$$F_z - \frac{d}{dx} F_p - \frac{d}{dy} F_q = 0. \qquad [13]$$

Equation (13) is the first necessary condition which $z = g(x, y)$ must satisfy in order that this surface render the integral I a minimum. The integral I is said to be stationary for $z = g(x, y)$ when Eq. (13) holds.

1·5. Generalization of the Simplest Case: More than One Dependent Variable. Of great importance in the calculus of variations from the viewpoint of applied mathematics is the minimizing of an integral whose integrand is a function of more than one dependent variable and their derivatives. Accordingly, let it be required to minimize the integral

$$I = \int_{t_1}^{t_2} F(x, y, z, \cdots x', y', z' \cdots)dt, \qquad [14]$$

where $x, y, z \cdots$ are functions of t. The number of dependent variables is finite. We proceed as in §1·2 by letting

$$
\begin{aligned}
x &= x_1(t) + \alpha_1\eta_1(t),\\
y &= y_1(t) + \alpha_2\eta_2(t), \qquad [15]\\
&\cdots\cdots\cdots
\end{aligned}
$$

where $\eta_1(t) = \delta x$, $\eta_2(t) = \delta y$, $\cdots$ and $x_1(t)$, $x_2(t)$, $\cdots$ are minimizing functions. When the values of $x, y, z \cdots$ from Eqs. (15) are substituted in Eq. (14) the integral I is evidently a function of $\alpha_1, \alpha_2, \alpha_3, \cdots$. From the elementary theory of maxima and minima it follows that for $I(\alpha_1, \alpha_2, \alpha_3, \cdots)$ to possess an extremum (i.e., maximum or minimum) at $\alpha_1 = \alpha_2 = \alpha_3 = \cdots \alpha_n = 0$, it is necessary that

$$
\left.\frac{\partial I}{\partial \alpha_1}\right|_{\alpha_i=0} = \left.\frac{\partial I}{\partial \alpha_2}\right|_{\alpha_i=0} = \left.\frac{\partial I}{\partial \alpha_n}\right|_{\alpha_i=0} = 0.
$$

Accordingly we have

$$
\begin{aligned}
I_{\alpha_1}(0) &= \int_{t_1}^{t_2} [F_x\eta_1(t) + F_{x'}\eta_1'(t)]dt = 0,\\
I_{\alpha_2}(0) &= \int_{t_1}^{t_2} [F_y\eta_2(t) + F_{y'}\eta_2'(t)]dt = 0,\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

where

$$
I_{\alpha_i}(0) = \left.\frac{\partial I}{\partial \alpha_i}\right|_{\alpha_i=0} = 0 \; (i = 1, 2, \cdots, n).
$$

Integrating by parts and applying the familiar reasoning of §§ 1·1–1·3 we have

$$
\begin{aligned}
&\int_{t_1}^{t_2} \left(F_x - \frac{d}{dt} F_{x'}\right)\eta_1(t)dt = 0,\\
&\int_{t_1}^{t_2} \left(F_y - \frac{d}{dt} F_{y'}\right)\eta_2(t)dt = 0,\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

or

$$
\begin{aligned}
F_x - \frac{d}{dt} F_{x'} &= 0,\\
F_y - \frac{d}{dt} F_{y'} &= 0. \qquad [16]\\
&\cdots\cdots\cdots
\end{aligned}
$$

Equations (16) are the Euler equations of the problem. These equations can be derived more quickly by the variation method, i.e., by setting $\delta I = 0$.

$$\delta I = \int_{t_1}^{t_2} (F_x\,\delta x + F_{x'}\,\delta x' + F_y\,\delta y + F_{y'}\,\delta y' + \cdots)dt = 0.$$

The usual integration by parts yields

$$\delta I = \int_{t_1}^{t_2} \left[\left(F_x - \frac{d}{dt} F_{x'}\right)\delta x + \left(F_y - \frac{d}{dt} F_{y'}\right)\delta y + \cdots\right] dt = 0.$$

Since δx, δy, $\cdots$ are arbitrary, it follows that $\delta I = 0$ only in case Eqs. (16) are valid.

EXERCISES AND PROBLEMS III

1. By means of Eq. (13) show that the partial differential equation of a minimum surface is $(1 + q^2)r + (1 + p^2)t - 2\,pqs = 0$, where

$$r = \frac{dp}{dx}, \quad t = \frac{dq}{dy}, \quad \text{and} \quad s = \frac{\partial^2 z}{\partial y\,\partial x}.$$

The integral to be minimized is

$$A = \iint_S (1 + p^2 + q^2)^{1/2}\,dy\,dx.$$

2. The difference of the kinetic and potential energies of a dynamical system (two-dimensional automobile, § 1·10) is

$$F = m\left[\frac{1}{2}(z'^2 + k^2\theta'^2) - \frac{g}{2e}(z^2 + l^2\theta^2)\right],$$

where m, g, l, and e are constants. Find the curve $z = z(t)$ and $\theta = \theta(t)$ which renders the integral

$$I = \int_{t_1}^{t_2} F(z, \theta, z', \theta')dt$$

stationary.

(2)

Hamilton's Principle

An understanding of the three simplest problems (§§1·2–1·5) in the calculus of variations leads naturally to Hamilton's principle. As previously noted, this principle is the most important single one in mathematical physics, since it holds not only for nearly [3] all dynami-

[3] Appell has shown that the constraints, if any, of the motion considered must be independent of the velocities.

cal systems, but is also valid in its applications to electrical phenomena, theory of elasticity, wave mechanics, and other divisions of engineering and physics.

1·6. Statement of Hamilton's Principle. One form of Hamilton's principle, stated in the language of the calculus of variations, is

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0, \qquad [17]$$

where T and V are respectively the kinetic and potential energies of a physical system and t_1 and t_2 are two instants of time during the motion of the system. In this form V is the negative of a function U such that the partial derivatives of U in any direction give the force in that direction. Equation (17) is the usual form in which Hamilton's principle is encountered. However, a more general statement of this principle is

$$\int_{t_1}^{t_2} [\delta T + \Sigma(X_i \delta x_i + Y_i \delta y_i + Z_i \delta z_i)]dt = 0, \qquad [18]$$

where T is the kinetic energy of the physical system; X_i, Y_i, Z_i are forces acting during the motion of the system and δx_i, δy_i, δz_i are variations of coordinates of the system.

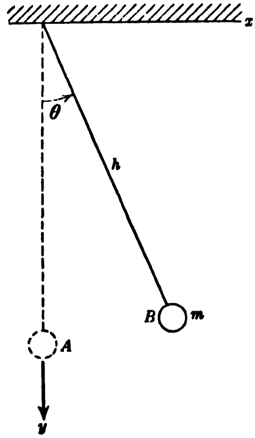

FIG. 1·4. Simple Pendulum.

Before proving this principle for dynamical systems we employ it in the solution of an elementary problem. Let it be required to write the differential equations of motion of the simple pendulum of Fig. 1·4. The kinetic energy of the system is $T = \frac{1}{2}m\, k^2\theta'^2$ where m is the mass and k is the radius of gyration. From the figure the potential energy evidently is

$$V = mgh(1 - \cos\theta). \qquad [19]$$

By Hamilton's principle we have

$$\delta \int_{t_1}^{t_2} [\tfrac{1}{2}m\, k^2\theta'^2 - mgh(1 - \cos\theta)]dt = 0,$$

or

$$\int_{t_1}^{t_2} [m\, k^2\theta'\, \delta\theta' - mgh \sin\theta\, \delta\theta]dt = 0,$$

or

$$\int_{t_1}^{t_2} [m\, k^2\theta' \frac{d}{dt}\, \delta\theta - mgh \sin\theta\, \delta\theta]dt = 0.$$

By application of the usual integration by parts and the familiar reasoning of § 1·2 the last equation becomes

$$\int_{t_1}^{t_2} m[gh \sin \theta + k^2\theta'']\delta\theta \, dt = 0,$$

or

$$k^2\theta'' = - gh \sin \theta.$$

This is the required equation.

To understand more fully Hamilton's principle, let a system of n particles experience a change of position according to Newton's laws

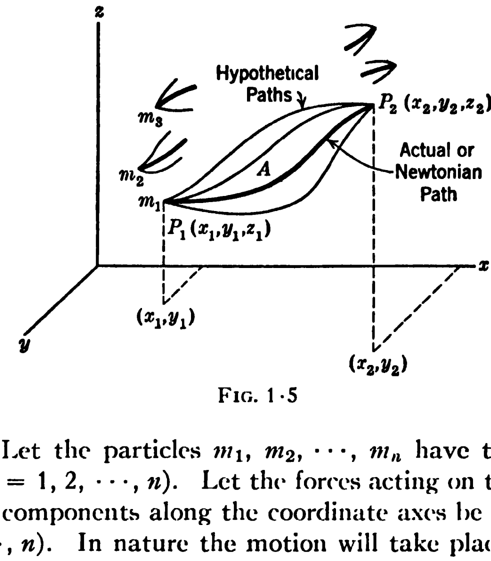

FIG. 1·5

of motion. Let the particles $m_1, m_2, \cdots, m_n$ have the coordinates (x_i, y_i, z_i), $(i = 1, 2, \cdots, n)$. Let the forces acting on the particles be F_i and their components along the coordinate axes be X_i, Y_i, and Z_i $(i = 1, 2, \cdots, n)$. In nature the motion will take place according to Newton's laws of motion, i.e.,

$$\begin{aligned} m_i x_i'' &= X_i, \\ m_i y_i'' &= Y_i, \\ m_i z_i'' &= Z_i. \end{aligned} \qquad [20]$$

To fix the ideas let the ith particle m_i be at P_1 at time t_1 and at P_2 at time t_2 and let the path described in the interval $t_2 - t_1$ be $P_1 A P_2$ in Fig. 1·5. Among all the mathematically possible paths which m_i might have described, is the actual one the most economical one in the sense that the integral

$$\int_{P_1}^{P_2} m_i v_i \, ds_i = \text{action} \qquad [21]$$

was a minimum? In the integrand v_i is the velocity of m_i and ds_i an element of distance. (The statement that the action as given by (21) is a minimum for the actual motion of a particle or system is known as the principle of least action.) It was the above question asked by Maupertius (1698–1759) which led eventually (150 years later) through the works of Lagrange, Jacobi, and Hamilton (1805–1865) to Hamilton's principle. Lagrange in 1788 and Jacobi later answered this question in the affirmative for certain types of motion. Hamilton encountered difficulty in understanding Lagrange's proof that the action is a minimum for the actual motion and derived instead the related principle which bears his name. It can be shown from Hamilton's principle that the actual motion, i.e , motion according to Newton's laws, which takes place between two points (positions) in the time $t_2 - t_1$ is such as to render the integral

$$\int_{t_1}^{t_2} (T - V)dt$$

a minimum when compared with any other infinitely near motion between the same two points provided the time interval is the same in both motions. Hamilton's principle does not state so much. It states that the actual motion renders the above integral stationary, i.e., that the first variation vanishes.

1·7. Proof of Hamilton's Principle. To see that Eq. (18) holds for the actual motion of the particle (or system), it is necessary to consider a field of paths near $P_1 A P_2$ (Fig. 1·5) in which the actual path or Newtonian path is imbedded. Motion according to Newton's laws can follow only the actual path. Consequently, all other paths are fictitious or hypothetical. Let $x_i = x_i(t)$, $y_i = y_i(t)$, $z_i = z_i(t)$ denote the Newtonian path and

$$\begin{aligned} x_i &= x_i(t) + \delta x_i(t), \\ y_i &= y_i(t) + \delta y_i(t), \\ z_i &= z_i(t) + \delta z_i(t) \end{aligned} \qquad [22]$$

be the neighboring hypothetical paths of the particle m_i where δx_i, δy_i, δz_i are arbitrary and independent but small variations of x_i, y_i, z_i. Hamilton's principle is now easily established.

The kinetic energy of the system is $T = 1/2\Sigma m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$ where the summation is taken over the n particles.[4] Multiply Eqs. (20) respectively by δx_i, δy_i, and δz_i and add. The result is

$$\Sigma m_i(\ddot{x}_i\,\delta x_i + \ddot{y}_i\,\delta y_i + \ddot{z}_i\,\delta z_i) = \Sigma(X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i). \quad [23]$$

[4] The notation $\dot{x}$ is frequently used for $\dfrac{dx}{dt}$.

It is easily verified by means of Eqs. (9–10) that

$$\ddot{x}_i\,\delta x_i = (\dot{x}_i\,\delta x_i)' - \dot{x}_i(\delta x_i)' = (\dot{x}_i\,\delta x_i)' - \tfrac{1}{2}\delta\dot{x}_i^2. \qquad [24]$$

Equation (23) may be written

$$\Sigma\{m_i[\delta\tfrac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - (\dot{x}_i\,\delta x_i + \dot{y}_i\,\delta y_i + \dot{z}_i\,\delta z_i)'] + (X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i)\} = 0. \qquad [25]$$

If Eq. (25) is integrated from t_1 to t_2 we have

$$\int_{t_1}^{t_2}[\delta T + \Sigma(X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i)]\,dt - \Sigma(\dot{x}_i\,\delta x_i + \dot{y}_i\,\delta x_i + \dot{z}_i\,\delta z_i)\Big|_{t_1}^{t_2} = 0.$$

The last term of the left member is zero since $\delta x_i = \delta y_i = \delta z_i = 0$ at $t = t_1$ and $t = t_2$. Finally then

$$\int_{t_1}^{t_2}[\delta T + \Sigma(X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i)]dt = 0. \qquad [26]$$

Equation (26) is the general form of Hamilton's principle, Eq. (18). If instead of a system of particles we have a continuous body, then

$$\Sigma\tfrac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = \tfrac{1}{2}\int v^2dm.$$

It should be noted that $\Sigma(X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i)$ is the work done in an infinitesimal displacement of the system *by* the forces X_i, Y_i, Z_i. It is not difficult to obtain Eq. (17) from Eq. (26).

1·8. Potential Energy of Dynamical Systems. A discussion of potential energy is of value before deriving Eq. (17) from Eq. (26).

The value of V in Eq. (19) is the potential energy of the pendulum in configuration B with respect to configuration A (Fig. 1·4). Potential energy is the amount of work done *against* gravity in bringing the pendulum from A to B. Likewise the potential energy of a system of bodies in a configuration B with respect to A is the work which must be done against the forces acting on the system of bodies to bring the system from A to B.

Let W be the work required to move a system of bodies from configuration A to configuration B *against* a system (or field) of forces F. Next let the system return to configuration A. If the work done *by* the forces F on the bodies during this return is also W then the system of forces and also the dynamical system are said to be **conservative.** If a system is not conservative it is called **dissipative.** If a system is con-

servative a function called the **potential** always exists and is defined to be the negative of the potential energy. It easily follows from the definition of a conservative field of force that the potential energy is independent of the path by which the system attained a given configuration.

An equivalent definition of the potential function is the following. If X, Y, Z are single-valued functions of x, y, z which do not contain t explicitly and if there exists a function $U(x, y, z)$ such that

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}, \tag{27}$$

then U is called the **potential function.** To see that this definition is equivalent to the definition of the potential function as the negative of the potential energy, multiply Eqs. (27) respectively by dx, dy, dz, and add. We then obtain

$$X\,dx + Y\,dy + Z\,dz = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz.$$

The left member of this equation is the work done *by* the forces in an elementary displacement. The right member is an exact differential dU. Consequently, the value of dU integrated along all paths from x_0, y_0, z_0 to x, y, z is the same and is the total work done. Thus

$$W = U(x, y, z) - U(x_0, y_0, z_0) = U(x, y, z) - \text{constant}.$$

U is thus the work done *by* the forces and $-U$ the work done *against* the forces or the potential energy.

1·9. Derivation of First Form of Hamilton's Principle. Equation (17) is now obtained from Eq. (18). The work done by the forces X, Y, Z in an infinitesimal displacement in the general form of Hamilton's principle is

$$X\,\delta x + Y\,\delta y + Z\,\delta z.$$

If a potential function U exists, this same work is $-dV$ or $-\delta V$. Consequently, substituting this value for the elementary work in (18) we obtain

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0.$$

1·10. Engineering Applications of Hamilton's Principle. Hamilton's principle is useful in deriving the differential equations of motion of holonomic [5] dynamical systems.

[5] Holonomic systems are defined in Sec. 3.

EXAMPLE 1. *Two-dimensional automobile.* A uniform beam of mass m and length $2l$ is supported on two equal springs as shown in Fig. 1·6, and such that the beam has but two degrees of freedom: one a small oscillation of the center of gravity in a vertical line, and the other a small rotation about a line through the center of gravity and perpendicular to the plane of the figure. Write the differential equations of motion.

Evidently, by Koenig's theorem,[6] the kinetic energy is $T = (m/2)(\dot{z}^2 + k^2\dot{\theta}^2)$, where m is the mass of the beam and k is its radius of gyration about the center of gravity of the beam. The potential energy consists of two parts, V_1 and V_2.

V_1 = work done against the springs and by gravity in a vertical displacement.

V_2 = work done by an angular displacement about the center of gravity.

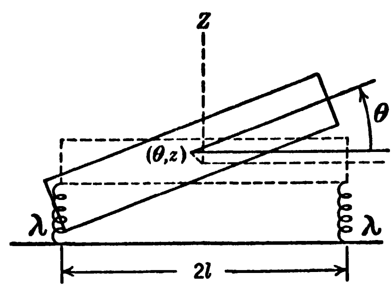

FIG. 1·6. Two-dimensional Automobile.

Let λ and e be respectively the spring constant and displacement of an end of the spring in equilibrium position under the force of gravity. Then since each spring bears half of the weight $\frac{mg}{2} = \lambda e$. If the beam is given a vertical displacement, the elementary work done is

$$dV_1 = -mg\,dz + 2\lambda(e + z)dz,$$

or

$$V_1 = -mgz + \lambda(e + z)^2 + C_1.$$

Taking V_1 to be zero in equilibrium position, i.e., $z = 0$, we find $C_1 = -\lambda e^2$. Remembering that $2\lambda e = mg$, we have

$$V_1 = \lambda z^2 = \frac{mg}{2e} z^2.$$

The work done in an infinitesimal rotation is

$$dV_2 = \lambda(e + x)dx - \lambda(e - x)dx$$

$$V_2 = \frac{\lambda}{2}(e + x)^2 + \frac{\lambda}{2}(e - x)^2 + C_2.$$

[6] See §1·17 for Koenig's theorem.

If $V_2 = 0$ for $x = 0$, then

$$V_2 = \lambda x^2 = \frac{mg}{2e} l^2\theta^2.$$

The total potential energy V is

$$V = V_1 + V_2 = \frac{mg}{2e} (z^2 + l^2\theta^2).$$

By applying Hamilton's principle we have

$$\delta \int_{t_1}^{t_2} \left[\tfrac{1}{2}m(\dot{z}^2 + k^2\dot{\theta}^2) - \frac{g}{2e} m(z^2 + l^2\theta^2) \right] dt$$

$$= \int_{t_1}^{t_2} \left[m(\dot{z}\, \delta\dot{z} + k^2\dot{\theta}\, \delta\dot{\theta}) - \frac{gm}{e} (z\, \delta z + l^2\theta\, \delta\theta) \right] dt$$

$$= \int_{t_1}^{t_2} m \left[(\dot{z}\, \delta\dot{z} - \frac{gz}{e}\, \delta z) + k^2\dot{\theta}\, \delta\dot{\theta} - \frac{g}{e} l^2\theta\, \delta\theta \right] dt = 0.$$

By the procedure of § 1.5 the last equation becomes

$$\int_{t_1}^{t_2} \left[\left(\ddot{z} + \frac{g}{e} z\right) \delta z + \left(k^2\ddot{\theta} + \frac{gl^2\theta}{e}\right) \delta\theta \right] dt = 0.$$

By the usual reasoning the differential equations are

$$\ddot{z} + \frac{gz}{e} = 0,$$

$$\ddot{\theta} + \frac{gl^2\theta}{k^2e} = 0.$$

EXAMPLE 2. *Simple accelerometer.* A simple accelerometer is constructed of a mass M, a spring S, and two identical carbon-piles

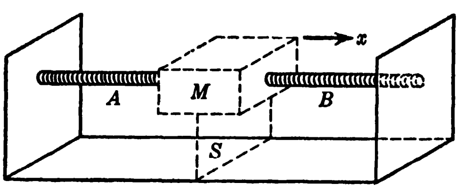

FIG. 1·7. Accelerometer.

A and B as shown in Fig. 1·7. The combined spring-constant of one carbon-pile and the spring S is λ. The mass M possesses one degree of freedom and the compression of each carbon-pile is e when M is in equilibrium position. Obtain the natural period of the instrument.

The elementary work done in displacing M is

$$dV = \lambda(e + x)dx - \lambda(e - x)dx,$$

whence

$$V = \frac{\lambda}{2}(e + x)^2 + \frac{\lambda}{2}(e - x)^2 + C.$$

Since

$$V = 0 \quad \text{for} \quad x = 0, \quad C = -\lambda e^2$$

and

$$V = \lambda x^2.$$

The kinetic enegy is $M\dot{x}^2/2$. Hamilton's principle,

$$\delta \int_{t_1}^{t_2} (M\dot{x}^2/2 - \lambda x^2)dt = 0,$$

yields the differential equation

$$M\ddot{x} + 2\lambda x = 0$$

whose general solution is

$$x = A \sin \sqrt{2\lambda/M}\, t + B \cos \sqrt{2\lambda/M}\, t.$$

The period of oscillation is $2\pi \sqrt{M/2\lambda}$.

EXAMPLE 3. *Simple seismograph.* A gate hung on an inclined support together with a recording device is a simple seismograph. Let

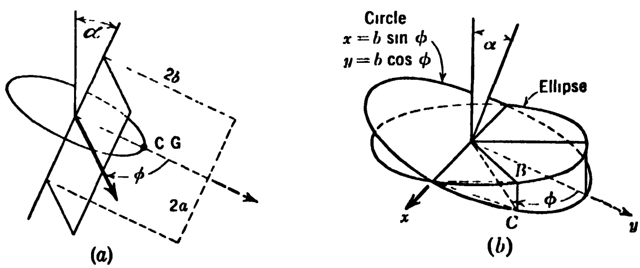

FIG. 1·8. Simple Seismograph.

the mass, length, width, and inclination be respectively, m, $2a$, $2b$, and α. Obtain the natural period of a small oscillation of the instrument.

The kinetic energy is $I\dot{\varphi}^2/2 = 2mb^2\dot{\varphi}^2/3$. The potential energy V for an angular displacement φ from equilibrium configuration is

$$\begin{aligned} V &= mg(b \sin \alpha - BC) \\ &= mg(b \sin \alpha - AB \tan \alpha) = mg(b \sin \alpha - y \cos \alpha \tan \alpha). \end{aligned}$$

From the equation of the circle $y = b\cos\varphi$, $x = b\sin\varphi$, the potential energy reduces to

$$V = mgb\sin\alpha(1 - \cos\varphi).$$

Hamilton's principle yields the differential equation

$$\ddot{\varphi} + \frac{3g}{4b}\sin\alpha\sin\varphi = 0.$$

For a small oscillation the last equation becomes

$$\ddot{\varphi} + \left(\frac{3g}{4b}\sin\alpha\right)\varphi = 0.$$

The period of oscillation is $4\pi\sqrt{\dfrac{b}{3g\sin\alpha}}$.

The large majority of the exercises and problems of the remaining eleven problem sets of this chapter reduces to systems of ordinary differential equations with constant coefficients. If it is desirable on the part of the instructor and students to solve completely each problem as soon as the differential equations are derived, then §§ 1·26 and 1·31 can be studied simultaneously with the material of §§ 1·11–1·26 and the solutions of the systems of differential equations obtained. The mathematical technique for solving many systems of differential equations which are non-linear or otherwise difficult is found in Chap. III.

EXERCISES IV

Solve the following exercises by means of Hamilton's principle.

1. Neglecting air resistance, obtain the differential equations of the motion of a projectile. Assume the projectile to be a particle of mass m.

2. A weight $4W$ is attached to a string which passes over a fixed pulley. The other end of the string is attached to a pulley of weight W. A second string, to which weights W and $2W$ have been fastened, passes over a second pulley as shown in Fig. 1·9. Write the differential equation of motion of the weight $4W$.

3. Obtain the differential equations of the oscillations of the double pendulum shown in Fig. 1·10. The bobs have masses m_1 and m_2 and the strings have lengths a and b. Assume there is no damping.

4. Three circular discs can oscillate only in horizontal planes as shown in Fig. 1·11. Their masses and radii are respectively m_1, m_2, m_3; r_1, r_2, r_3. The torque coefficients of the rods are k_1, k_2, k_3. Obtain the differential equations of motion. Assume no damping.

5. Three uniform steel discs are mounted on a horizontal shaft as shown in Fig 1·12. The radii and weights of the discs are respectively $r_1 = 40$ in., $r_2 = 10$ in., $r_3 = 40$ in.; $W_1 = 4000$ lb., $W_2 = 1000$ lb., $W_3 = 4000$ lb. The torque coefficients are $k_1 = k_2 = 30 \times 10^6$. Write the differential equations of the free torsional oscillations of the discs.

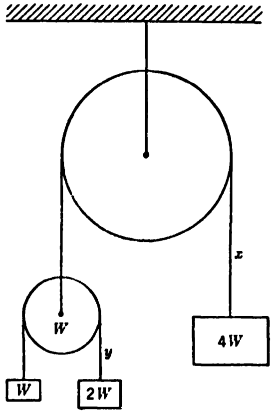

FIG. 1·9

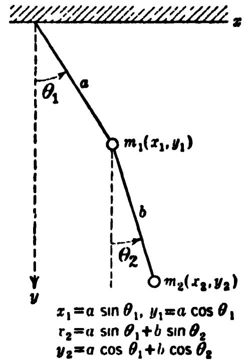

FIG. 1·10

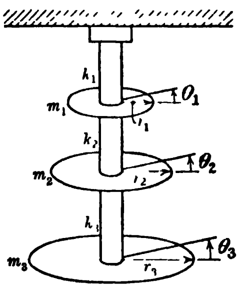

FIG. 1·11

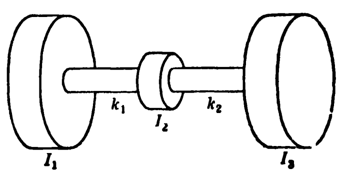

FIG. 1·12

(3)

Lagrange's Equations

Lagrange's equations are readily derived from Hamilton's principle. These equations are a system of n simultaneous differential equations whose dependent variables are the n coordinates $q_1, q_2, \cdots, q_n$ specifying the configuration of a dynamical system at any time.

1·11. Generalized Coordinates; Holonomic Systems. If the constitution of a dynamical system is given, its configuration can be specified by means of a definite number of quantities which vary when its configuration is changed. These quantities (denoted $q_1, q_2, \cdots, q_n$) are called **generalized coordinates** because of their general nature. In example 1, § 1·10, the configuration of the beam is specified, subject to the restricted motion designated, by the two generalized coordinates

z and θ. The first coordinate is a linear displacement, the second an angular displacement. In exercise 4, § 1·10, the coordinates are the three angles θ_1, θ_2, θ_3. Generalized coordinates may also be volumes, charges, currents, etc. A system is said to possess **n degrees of freedom** if the *least* number of generalized coordinates necessary to specify a general position of the system is n.

Dynamical systems are divided, for analytical treatment, into two classes, holonomic and non-holonomic. If a configuration of a system is specified by the n generalized coordinates $q_1, q_2, \cdots, q_n$ and an adjacent configuration is specified by $q_1 + \delta q_1, q_2 + \delta q_2, \cdots, q_n + \delta q_n$ where $\delta q_1, \delta q_2, \cdots, \delta q_n$ are *arbitrarily independent* infinitesimal quantities then the system is said to be **holonomic.** For the holonomic systems of the present section, n denotes the number of degrees of freedom. As a simple example of a holonomic system consider a sphere to move on a smooth plane. The sphere, since the plane is smooth, can both roll and slide. The quantities giving the position and orientation of the sphere are x, y, the coordinates of the point of contact with the plane and θ, φ, ψ, the orientation of the sphere about its center. In an adjacent position the coordinates are $x + \delta x$, $y + \delta y$, $\theta + \delta\theta$, $\varphi + \delta\varphi$, $\psi + \delta\psi$, and the increments δx, δy, $\delta\theta$, $\delta\varphi$, $\delta\psi$ of the coordinates are all mutually independent.

In the case of a non-holonomic system the increments $\delta q_1, \delta q_2, \cdots, \delta q_n$ of the n generalized coordinates $q_1, q_2, \cdots, q_n$ are not independent. There exist constraints on the system which are expressible in the form of the $m < n$ equations

$$
\begin{aligned}
C_{11}\, dq_1 + C_{12}\, dq_2 + \cdots + C_{1n}\, dq_n + T_1\, dt &= 0,\\
C_{21}\, dq_1 + C_{22}\, dq_2 + \cdots + C_{2n}\, dq_n + T_2\, dt &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
C_{m1}\, dq_1 + C_{m2}\, dq_2 + \cdots + C_{mn}\, dq_n + T_m\, dt &= 0,
\end{aligned}
\qquad [28]
$$

where the C's and T's are, in general, functions of $q_1, q_2, \cdots, q_n, t$ and where (28) are *non-integrable* equations. As an example, suppose a sphere, resident on a *rough* plane, to move so that x and y change simultaneously. Since sliding is impossible, non-integrable relations must hold between the two increments δx, δy and the increments $\delta\theta$, $\delta\varphi$, $\delta\psi$.

1·12. Lagrange's Equations. A rigid body is an aggregation of particles connected in such a way that their mutual distances are invariable. A dynamical system is regarded as a number of particles or bodies subject to connections and constraints.

Let a system consist of n particles. Suppose the coordinates of the ith particle, whose mass is m_i, of the system to be related to the n coordinates $q_1, q_2, \cdots, q_n$ by the relations

$$\begin{aligned} x_i &= f_i(q_1, q_2, \cdots, q_n; t), \\ y_i &= g_i(q_1, q_2, \cdots, q_n; t), \\ z_i &= h_i(q_1, q_2, \cdots, q_n; t). \end{aligned} \tag{29}$$

The kinetic energy T of the system is, by definition,

$$T = \tfrac{1}{2}\Sigma m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2),$$

where the summation extends over all particles of the system. The total derivatives with respect to the time of Eqs. (29) are

$$\begin{aligned} \dot{x}_i &= \frac{\partial f_i}{\partial q_1}\dot{q}_1 + \frac{\partial f_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial f_i}{\partial q_n}\dot{q}_n + \frac{\partial f_i}{\partial t}, \\ \dot{y}_i &= \frac{\partial g_i}{\partial q_1}\dot{q}_1 + \frac{\partial g_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial g_i}{\partial q_n}\dot{q}_n + \frac{\partial g_i}{\partial t}, \\ \dot{z}_i &= \frac{\partial h_i}{\partial q_1}\dot{q}_1 + \frac{\partial h_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial h_i}{\partial q_n}\dot{q}_n + \frac{\partial h_i}{\partial t}, \end{aligned}$$

and hence

$$\begin{aligned} T = \tfrac{1}{2}\sum m_i\Bigg[&\left(\frac{\partial f_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_i}{\partial q_n}\dot{q}_n + \frac{\partial f_i}{\partial t}\right)^2 \\ &+ \left(\frac{\partial g_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial g_i}{\partial q_n}\dot{q}_n + \frac{\partial g_i}{\partial t}\right)^2 \\ &+ \left(\frac{\partial h_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial h_i}{\partial q_n}\dot{q}_n + \frac{\partial h_i}{\partial t}\right)^2\Bigg] \\ = T(q_1, q_2, &\cdots, q_n; \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n; t). \end{aligned}$$

Thus the kinetic energy is a function of the coordinates, their derivatives, and the time.

The expression $\Sigma(X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i)$ in Hamilton's principle (see Eq. 26) is the total work done on the particles of the system *by* the forces acting on the system. It is required to express this work as a function of the generalized coordinates and their increments. The variations of Eqs. (29) are

$$\delta x_i = \frac{\partial f_i}{\partial q_1}\delta q_1 + \frac{\partial f_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f_i}{\partial q_n}\delta q_n,$$

$$\delta y_i = \frac{\partial g_i}{\partial q_1}\,\delta q_1 + \frac{\partial g_i}{\partial q_2}\,\delta q_2 + \cdots + \frac{\partial g_i}{\partial q_n}\,\delta q_n,$$

$$\delta z_i = \frac{\partial h_i}{\partial q_1}\,\delta q_1 + \frac{\partial h_i}{\partial q_2}\,\delta q_2 + \cdots + \frac{\partial h_i}{\partial q_n}\,\delta q_n.$$

(The variation $\delta t = 0$; see § 1·3.) Substituting δx_i, δy_i, δz_i in the expression for the total work we have

$$\sum (X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i) =$$

$$\sum \left[\left(X_i \frac{\partial f_i}{\partial q_1} + Y_i \frac{\partial g_i}{\partial q_1} + Z_i \frac{\partial h_i}{\partial q_1}\right)\delta q_1 + \left(X_i \frac{\partial f_i}{\partial q_2} + Y_i \frac{\partial g_i}{\partial q_2} + Z_i \frac{\partial h_i}{\partial q_2}\right)\delta q_2 \right.$$
$$\left. + \cdots + \left(X_i \frac{\partial f_i}{\partial q_n} + Y_i \frac{\partial g_i}{\partial q_n} + Z_i \frac{\partial h_i}{\partial q_n}\right)\delta q_n\right].$$

Finally, substituting $T(q_1, q_2, \cdots, q_n, \dot q_1, \dot q_2, \cdots, \dot q_n, t)$ and the above expression for the total work in Eq. (26) we obtain

$$\int_{t_1}^{t_2}\left\{\delta T + \sum\left[\left(X_i \frac{\partial f_i}{\partial q_1} + Y_i \frac{\partial g_i}{\partial q_1} + Z_i \frac{\partial h_i}{\partial q_n}\right)\delta q_1 + \cdots \right.\right.$$
$$\left.\left. + \left(X_i \frac{\partial f_i}{\partial q_n} + Y_i \frac{\partial g_i}{\partial q_n} + Z_i \frac{\partial h_i}{\partial q_n}\right)\delta q_n\right]\right\} dt = 0.$$

By taking the variation of T with respect to $q_1, q_2, \cdots, q_n, \dot q_1, \dot q_2, \cdots, \dot q_n$ the last equation becomes

$$\int_{t_1}^{t_2}\left\{\frac{\partial T}{\partial \dot q_1}\,\delta \dot q_1 + \cdots + \frac{\partial T}{\partial \dot q_n}\,\delta \dot q_n + \frac{\partial T}{\partial q_1}\,\delta q_1 + \cdots + \frac{\partial T}{\partial q_n}\,\delta q_n \right.$$
$$+ \sum\left(X_i \frac{\partial f_i}{\partial q_1} + Y_i \frac{\partial g_i}{\partial q_1} + Z_i \frac{\partial h_i}{\partial q_1}\right)\delta q_1 + \cdots$$
$$\left. + \left(X_i \frac{\partial f_i}{\partial q_n} + Y_i \frac{\partial g_i}{\partial q_n} + Z_i \frac{\partial h_i}{\partial q_n}\right)\delta q_n\right\} dt = 0. \qquad [30]$$

Performing the usual integration by parts and applying the reasoning following Eq. (11), since $\delta q_1, \delta q_2, \cdots, \delta q_n$ are arbitrary, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_1}\right) - \frac{\partial T}{\partial q_1} = Q_1,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_2}\right) - \frac{\partial T}{\partial q_2} = Q_2, \qquad [31]$$
$$\cdots\cdots\cdots$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_n}\right) - \frac{\partial T}{\partial q_n} = Q_n,$$

where

$$Q_1 = \sum \left(X_i \frac{\partial f_i}{\partial q_1} + Y_i \frac{\partial g_i}{\partial q_1} + Z_i \frac{\partial h_i}{\partial q_1} \right),$$

$$Q_2 = \sum \left(X_i \frac{\partial f_i}{\partial q_2} + Y_i \frac{\partial g_i}{\partial q_2} + Z_i \frac{\partial h_i}{\partial q_2} \right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q_n = \sum \left(X_i \frac{\partial f_i}{\partial q_n} + Y_i \frac{\partial g_i}{\partial q_n} + Z_i \frac{\partial h_i}{\partial q_n} \right).$$

The $Q_1, Q_2, \cdots, Q_n$ are called **generalized forces.** Equations (31) are the equations of Lagrange for holonomic systems. The above derivation is both for systems and for a single particle.

If the motion of a system is described by s equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, \cdots, s)$$

plus the $m(m < s)$ Eqs. (28), where (28) are *integrable*, then the system is *holonomic* and possesses $s - m = n$ degrees of freedom. In all systems of Sec. 3, Eqs. (28) are suppressed (i.e., $m = 0$) by choice of coordinate systems such that the number of q's introduced equals the number of degrees of freedom.

Equations (28) are employed in both Secs. 5 and 6.

The Q's are not necessarily forces, but are quantities such that $Q_r\, \delta q_r$ is the work done by the forces acting on the system during the displacement δq_r while all other displacements of the system are zero. For example, if δq_r is

(*a*) a linear displacement, Q_r is a force,
(*b*) an angular displacement, Q_r is a moment,
(*c*) an increase in volume, Q_r is pressure.

If all of the forces acting on the system possess a potential function U, which is the negative of the potential energy V (see § 1·8), then $Q_r = -\dfrac{\partial V}{\partial q_r}$, $r = 1, 2, \cdots, n$. If some of the forces acting on the system (for example the forces due to springs on which the system is mounted) possess a potential then

$$Q_r = -\frac{\partial V}{\partial q_r} + Q_r^{(0)},$$

where $Q_r^{(0)}$ are those independent additional forces which are not contained in $-\dfrac{\partial V}{\partial q_r}$.

1·13. Illustrative Examples. The kinetic and potential energies and generalized forces in the following problems are computed by means of elementary principles of mechanics.

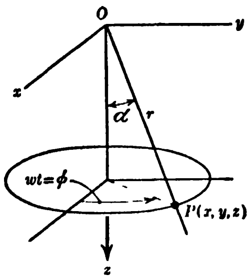

FIG. 1 13

EXAMPLE 1. A particle of mass m moves without friction on a straight line inclined at an angle α with the vertical. The line rotates about the vertical line with constant angular velocity w. Find the equation of motion.

The position of the particle m is given at time t by

$$x = r \sin \alpha \cos wt,$$

$$y = r \sin \alpha \sin wt,$$

$$z = r \cos \alpha,$$

where r is the variable distance of m from O. The kinetic and potential energies are respectively

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m(\dot{r}^2 + w^2 r^2 \sin^2 \alpha),$$

$$V = \text{constant} - mgz = \text{constant} - mgr \cos \alpha.$$

Since

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial T}{\partial r} = mw^2 r \sin^2 \alpha, \quad \frac{\partial V}{\partial r} = - mg \cos \alpha$$

substitution of these quantities in Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = - \frac{\partial V}{\partial r}$$

gives the differential equation

$$\ddot{r} - (w \sin \alpha)^2 r = g \cos \alpha$$

whose solution is

$$r = Ae^{wt \sin \alpha} + Be^{-wt \sin \alpha} - \frac{g \cos \alpha}{w^2 \sin^2 \alpha}.$$

EXAMPLE 2. A particle moving in free space is subject to a force $\mathbf{F}(r, \theta, \varphi)$, where r, θ, φ are spherical coordinates of its position. The coordinates r, θ, φ are related to rectangular coordinates x, y, z by the relations

$$x = r \cos \varphi \cos \theta,$$

$$y = r \cos \varphi \sin \theta,$$

$$z = r \sin \varphi.$$

The kinetic energy is

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m[\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\dot{\theta}^2 \cos^2 \varphi].$$

Substituting in Eqs. (31) we obtain

$$m[\ddot{r} - r(\dot{\varphi}^2 + \dot{\theta}^2 \cos^2 \varphi)] = Q_1,$$

$$m[(r^2\dot{\varphi})' + \dot{\theta}^2 r^2 \sin \varphi \cos \varphi] = Q_2,$$

$$m[(r^2\dot{\theta} \cos^2 \varphi)'] = Q_3.$$

Let **F** be resolved into the perpendicular components:

$F_r(r, \theta, \varphi)$ along r,

$F_\varphi(r, \theta, \varphi)$ perpendicular to r and in the meridian plane,

$F_\theta(r, \theta, \varphi)$ perpendicular to r and $F_\varphi(r, \theta, \varphi)$.

Then

$$Q_1 = F_r$$

$$Q_2 = rF_\varphi$$

$$Q_3 = rF_\theta \cos \varphi.$$

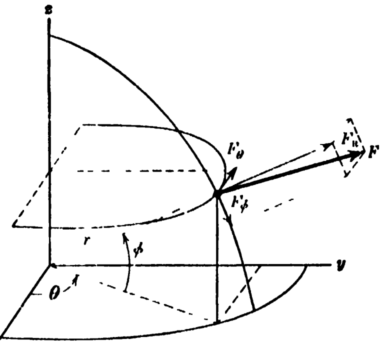

FIG. 1·14

EXAMPLE 3. A uniform rod of length $2l$ and mass m is free to slide without friction in a holder of negligible mass. The axis AB of the holder is inclined at an angle α to the vertical. The rod is turned until it is nearly horizontal and released. Assuming no friction obtain the differential equations of motion of the rod up to the time it leaves the holder.

Koenig's theorem [7] is useful in computing the kinetic energy of a rigid body. This theorem is: *the total kinetic energy of a rigid body of mass M consists of two parts: (a) the kinetic energy of a particle of mass M moving with the center of gravity of the body; (b) the kinetic energy of motion relative to the center of gravity, considered as fixed.*

Let the angular displacement of the holder from equilibrium position be θ (Fig. 1·15.) The first and second parts of the kinetic energy, as described in Koenig's theorem, are respectively

$$T_f = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

$$T_s = \tfrac{1}{2}mk^2\dot{\theta}^2,$$

[7] See §1·17 for proof of Koenig's theorem.

where k is the radius of gyration of the rod about the line $A'B'$. (The angular rotation of the rod about $A'B'$ is the same as the rotation about AB.) The total kinetic energy is

$$T = \tfrac{1}{2}m[\dot{r}^2 + (r^2 + k^2)\dot{\theta}^2].$$

The potential energy of the rod is

$$V = mgr(1 - \cos\theta)\sin\alpha. \quad \text{(See example 3, § 1·10.)}$$

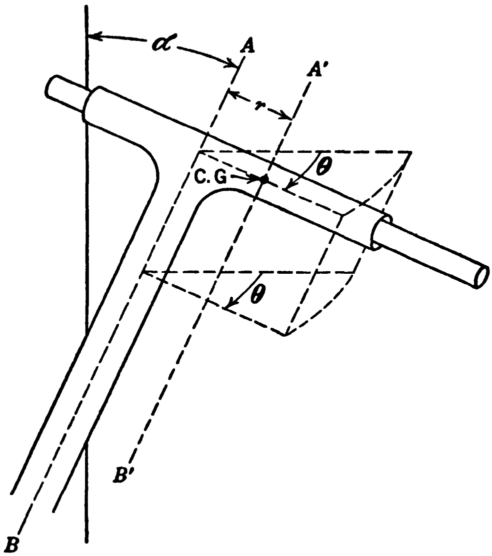

FIG. 1·15

Substituting T and V in Eqs. (31) we have the differential equations

$$\ddot{r} - r\dot{\theta}^2 + g(1 - \cos\theta)\sin\alpha = 0,$$

$$(r^2 + k^2)\ddot{\theta} + 2r\dot{r}\dot{\theta} + rg\sin\theta\sin\alpha = 0.$$

EXERCISES AND PROBLEMS V

1. Solve, by means of Lagrange's equations, the five exercises of §1·10.

1·14. Systems Subject to Dissipation Forces Proportional to Velocities. Suppose that there act on a system external resisting forces opposing the motion of the system and that each force is proportional to the first power of the velocity of its point of application. Then there exists a function,

$$F = \tfrac{1}{2}\Sigma(\alpha_i\dot{x}_i^2 + \beta_i\dot{y}_i^2 + \gamma_i\dot{z}_i^2)$$

called the Rayleigh dissipation function, such that Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r} = Q_r \quad (r = 1, 2, \cdots, n). \qquad [32]$$

Equations (32) are established as follows. Let the relations between x_ι, y_ι, z_ι and $q_1, q_2, \cdots, q_n$ be given by Eqs. (29). Let the work done *against* the motion of the system, i.e., the energy lost be written $\Sigma(\alpha_\iota \dot{x}_\iota\, \delta x_\iota + \beta_\iota \dot{y}_\iota\, \delta y_\iota + \gamma_\iota \dot{z}_\iota\, \delta z_\iota)$, where the summation is over all particles of the system. Then Hamilton's principle for the system is

$$\int_{t_1}^{t_2} [\delta T + \Sigma(X_\iota \delta x_\iota + Y_\iota\, \delta y_\iota + Z_\iota\, \delta z_\iota - \alpha_\iota \dot{x}_\iota\, \delta x_\iota - \beta_\iota \dot{y}_\iota\, \delta y_\iota - \gamma_\iota \dot{z}_\iota\, \delta z_\iota)]dt = 0, \qquad [33]$$

where X_ι, Y_ι, Z_ι are the components of all forces other than the dissipation forces. In view of (31) it is necessary to examine only the expression $-\Sigma(\alpha_\iota \dot{x}_\iota\, \delta x_\iota + \beta_\iota \dot{y}_\iota\, \delta y_i + \gamma_\iota \dot{z}_\iota\, \delta z_\iota)$. Recalling from § 1·12 the expressions for $\dot{x}_\iota$, $\dot{y}_\iota$, $\dot{z}_\iota$, δx_ι, δy_ι, and δz_ι we have

$$\begin{aligned}
&-\Sigma(\alpha_\iota \dot{x}_\iota\, \delta x_\iota + \beta_\iota \dot{y}_\iota\, \delta y_\iota + \gamma_\iota \dot{z}_\iota\, \delta z_\iota) \\
&= -\Bigg[\sum \alpha_\iota \left(\frac{\partial f_\iota}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_\iota}{\partial q_n}\dot{q}_n + \frac{\partial f_\iota}{\partial t}\right)\left(\frac{\partial f_\iota}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_\iota}{\partial q_n}\delta q_n\right) \\
&\quad + \sum \beta_\iota \left(\frac{\partial g_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial g_\iota}{\partial q_n}\dot{q}_n + \frac{\partial g_\iota}{\partial t}\right)\left(\frac{\partial g_\iota}{\partial q_1}\delta q_1 + \cdots + \frac{\partial g_\iota}{\partial q_n}\delta q_n\right) \\
&\quad + \sum \gamma_\iota \left(\frac{\partial h_\iota}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial h_\iota}{\partial q_n}\dot{q}_n + \frac{\partial h_\iota}{\partial t}\right)\left(\frac{\partial h_\iota}{\partial q_1}\delta q_1 + \cdots + \frac{\partial h_\iota}{\partial q_n}\delta q_n\right)\Bigg], \\
&= -\Bigg\{\sum \Bigg[\alpha_\iota \left(\frac{\partial f_\iota}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_\iota}{\partial q_n}\dot{q}_n + \frac{\partial f_\iota}{\partial t}\right)\frac{\partial f_i}{\partial q_1} \\
&\quad + \beta_i \left(\frac{\partial g_\iota}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial g_\iota}{\partial q_n}\dot{q}_n + \frac{\partial g_\iota}{\partial t}\right)\frac{\partial g_\iota}{\partial q_1} \\
&\quad + \gamma_\iota \left(\frac{\partial h_\iota}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial h_i}{\partial q_n}\dot{q}_n + \frac{\partial h_\iota}{\partial t}\right)\frac{\partial h_\iota}{\partial q_1}\Bigg]\delta q_1 \\
&\quad + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot + \\
&\qquad \sum \Bigg[\alpha_i \left(\frac{\partial f_\iota}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial f_i}{\partial q_n}\dot{q}_n + \frac{\partial f_\iota}{\partial t}\right)\frac{\partial f_\iota}{\partial q_n} \\
&\quad + \beta_\iota \left(\frac{\partial g_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial g_i}{\partial q_n}\dot{q}_n + \frac{\partial g_\iota}{\partial t}\right)\frac{\partial g_i}{\partial q_n} \\
&\quad + \gamma_\iota \left(\frac{\partial h_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial h_\iota}{\partial q_n}\dot{q}_n + \frac{\partial h_\iota}{\partial t}\right)\frac{\partial h_\iota}{\partial q_n}\Bigg]\delta q_n\Bigg\}, \\
&= -\left(\frac{\partial F}{\partial \dot{q}_1}\delta q_1 + \frac{\partial F}{\partial \dot{q}_2}\delta q_2 + \cdots + \frac{\partial F}{\partial \dot{q}_n}\delta q_n\right). \qquad [34]
\end{aligned}$$

To justify the last equal sign in (34) substitute the values of x_i, y_i, and z_i in F and compute the partial derivative of F with respect to $\dot{q}_r$ obtaining

$$\frac{\partial F}{\partial \dot{q}_r} = \sum \left[\alpha_i \left(\frac{\partial f_i}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial f_i}{\partial q_n} \dot{q}_n + \frac{\partial f_i}{\partial t} \right) \frac{\partial f_i}{\partial q_r} \right.$$
$$+ \beta_i \left(\frac{\partial g_i}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial g_i}{\partial q_n} \dot{q}_n + \frac{\partial g_i}{\partial t} \right) \frac{\partial g_i}{\partial q_r}$$
$$\left. + \gamma_i \left(\frac{\partial h_i}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial h_i}{\partial q_n} \dot{q}_n + \frac{\partial h_i}{\partial t} \right) \frac{\partial h_i}{\partial q_r} \right].$$

Substituting in (33) the value of $-\Sigma(\alpha_i \dot{x}_i \, \delta x_i + \beta_i \dot{y}_i \, \delta y_i + \gamma_i \dot{z}_i \, \delta z_i)$ obtained in Eqs. (34) and proceeding as in § 1·12 we obtain Eqs. (32).

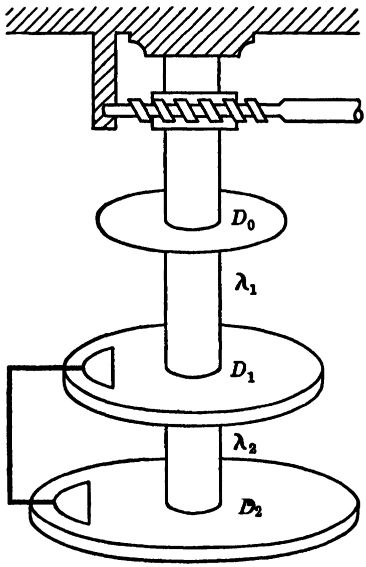

FIG. 1·16

EXAMPLE. It is desired to obtain uniform rotational motion by means of three heavy discs D_0, D_1, and D_2 suspended as indicated in Fig. 1·16. Disc D_0 is driven by means of a worm gear at as near uniform speed as possible. It is desired that D_2 rotate at a more uniform angular velocity than D_0. The discs are connected by thin rods of torque constants λ_1 and λ_2. Damping is effected between D_1 and D_2 by vanes immersed in fluid. Find the differential equations of motion of D_1 and D_2.

Let the angular displacement of D_2, D_1, and D_0 be θ_2, θ_1, and $\theta_0 = \omega_0 t + \Sigma(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$, where the Fourier series represents the variation in velocity due to imperfections of the gear, and ω_0 is the average angular velocity. If I_1 and I_2 are the moments of inertia of D_1 and D_2 the kinetic energy is

$$T = \tfrac{1}{2}(I_1 \dot{\theta}_1^2 + I_2 \dot{\theta}_2^2).$$

The potential energy is

$$V = \tfrac{1}{2}[\lambda_1(\theta_0 - \theta_1)^2 + \lambda_2(\theta_1 - \theta_2)^2].$$

The dissipation function for the relative velocities is

$$F = \tfrac{1}{2}\alpha(\dot{\theta}_1 - \dot{\theta}_2)^2.$$

Substitution in Eqs. (32) gives

$$I_1\ddot{\theta}_1 + \alpha\dot{\theta}_1 + (\lambda_1 + \lambda_2)\theta_1 - \alpha\dot{\theta}_2 - \lambda_2\theta_2 = \lambda_1\theta_0$$

$$-\alpha\dot{\theta}_1 - \lambda_2\theta_1 + I_2\ddot{\theta}_2 + \alpha\dot{\theta}_2 + \lambda_2\theta_2 = 0.$$

EXERCISES AND PROBLEMS VI

1. (Dynamic Vibration Absorber) A synchronous generator is driven by an engine which produces a component of pulsating torque $T \sin \omega t$. The distribution of mass of the rotating parts and the torque constant of the coupling shaft are such that there

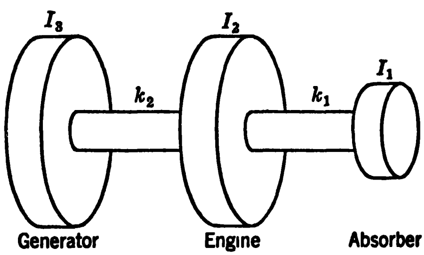

FIG. 1·17. Dynamic Vibration Absorber.

exist forced torsional vibrations of the rotor of the generator and the flywheel of the engine. These undesirable forced vibrations can be eliminated (or at least greatly diminished in amplitude) by what is known as a **dynamic vibration absorber.** In the present system this consists of extending a shaft in line with the coupling shaft and attaching thereto a disc I_1 as indicated in Fig. 1·17. If k_1, the torque constant, and I_1, the moment of inertia of the absorber, possess the proper values relative to the torque constant of the coupling shaft and the moments of inertia of the rotating parts of the generator and engine, the undesirable vibrations are eliminated.

Obtain the three differential equations of motion of the free torsional vibrations of the system shown in Fig. 1·17. The solution of these equations is reserved for problem set XI.

2. (Relative Damping) If the resisting forces acting on two particles m_1 (x_1, y_1, z_1) and m_2 (x_2, y_2, z_2) are

$$-k_1(\dot{x}_1 - \dot{x}_2),\ -k_2(\dot{y}_1 - \dot{y}_2),\ -k_3(\dot{z}_1 - \dot{z}_2)$$

and

$$-k_1(\dot{x}_2 - \dot{x}_1),\ -k_2(\dot{y}_2 - \dot{y}_1),\ -k_3(\dot{z}_2 - \dot{z}_1),$$

then the dissipation function is

$$\tfrac{1}{2}[k_1(\dot{x}_1 - \dot{x}_2)^2 + k_2(\dot{y}_1 - \dot{y}_2)^2 + k_3(\dot{z}_1 - \dot{z}_2)^2].$$

Deduce Lagrange's equations corresponding to Eqs. (32) for relative damping, i.e., for the case where the dissipation forces are proportional to the differences of the velocities of the points of application.

3. (Damped Dynamic Vibration Absorber) For machines which operate at one speed only, such as the synchronous generator of problem 1, the dynamic vibration absorber can be tuned sharply to operate at one frequency. In machines in which broad tuning is necessary, damping may be required in the system. Accordingly, let a damping device be introduced which acts on the coupling shaft between the engine and absorber. (Fig. 1·17.) Let the damping be relative damping and proportional to the difference between angular velocities of engine and absorber.

Write the differential equations of motion of the free torsional vibrations of the system. (Solution is required in problem set XI.)

1·15. Energies of Systems Possessing Several Degrees of Freedom. The kinetic energies of the systems thus far analyzed in this chapter have been easily obtained because the motions have been, for the most part, either motions of particles or the rotations of rigid bodies about fixed axes. Likewise, the potential energies of these systems have been found with little effort by the simple principle of elementary work. If, however, a rigid body possesses six degrees of freedom (three of translation and three of rotation) and if in addition the body is in any way connected to similar bodies, the calculation of the energies is difficult by the elementary methods employed thus far. For these more complicated motions recourse is had to vectors. This use of vectors eliminates all difficult visualization of relative motions and confusing projections of velocities. In § 1·16 sufficient formal theory of vectors is developed to render the calculation of energies a routine process.

1·16. Addition, Multiplication, Line Integrals, and Differentiation of Vectors. A **vector** is a quantity which possesses direction as well as magnitude; a **scalar** is a quantity which possesses magnitude only. Vector algebra is similar to scalar algebra. Zero and unit vectors are those whose magnitudes are respectively zero and one. Two vectors are equal, if and only if, they have the same magnitude and direction. By the negative vector $-\mathbf{A}$, we mean $\mathbf{A}$ with its direction reversed but its magnitude unchanged. A vector $\mathbf{A}$ can always be considered as $A\mathbf{a}$, where $\mathbf{a}$ is a unit vector and A is the magnitude of $\mathbf{A}$.

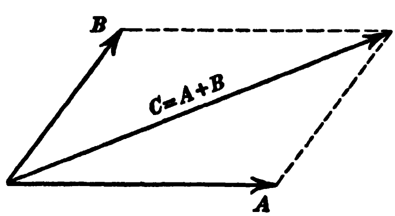

FIG. 1·18

(*a*) *Addition and subtraction.* $\mathbf{C}$, the sum of $\mathbf{A}$ and $\mathbf{B}$, is defined as the vector obtained by placing the initial point of $\mathbf{B}$ in coincidence with the terminal of $\mathbf{A}$ and taking $\mathbf{C}$ with its initial point coinciding with that of $\mathbf{A}$, and its terminal point with that of $\mathbf{B}$. From Fig. 1·18, evidently $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. The sum of three vectors $\mathbf{E} = \mathbf{A} + \mathbf{B} + \mathbf{D}$

$= \mathbf{C} + \mathbf{D}$, where $\mathbf{A} + \mathbf{B} = \mathbf{C}$. The subtraction of $\mathbf{A}$ is defined as the addition of $-\mathbf{A}$.

(*b*) *Vector components.* A vector is uniquely determined by giving its projections on the three coordinate axes. These projections are $A_x = A \cos(Ax)$, $A_y = A \cos(Ay)$ and $A_z = A \cos(Az)$, where (Ax) denotes the angle between the positive x-axis and $\mathbf{A}$. If $\mathbf{A} + \mathbf{B} = \mathbf{C}$, it is apparent geometrically that

$$A_x + B_x = C_x,$$
$$A_y + B_y = C_y,$$
$$A_z + B_z = C_z.$$

Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be unit vectors coinciding with the x, y, and z axes respectively. By the definition of addition

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

(*c*) *Scalar and vector products.* The scalar product of $\mathbf{A}$ by $\mathbf{B}$ (or $\mathbf{B}$ by $\mathbf{A}$) is a scalar defined by the equation $\mathbf{A}\cdot\mathbf{B} = AB \cos\theta$, where θ is the angle between the positive directions of A and B. The scalar product is thus the product of one vector by the projection of the other vector upon it. Hence $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$. Also

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \text{and} \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

It can be shown that $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$; thus we may write

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)\cdot(\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) \\
&= \mathbf{i}\cdot\mathbf{i}\, A_xB_x + \mathbf{i}\cdot\mathbf{j}\, A_xB_y + \mathbf{i}\cdot\mathbf{k}\, A_xB_z \\
&\quad + \mathbf{j}\cdot\mathbf{i}\, A_yB_x + \mathbf{j}\cdot\mathbf{j}\, A_yB_y + \mathbf{j}\cdot\mathbf{k}\, A_yB_z \\
&\quad + \mathbf{k}\cdot\mathbf{i}\, A_zB_x + \mathbf{k}\cdot\mathbf{j}\, A_zB_y + \mathbf{k}\cdot\mathbf{k}\, A_zB_z \\
&= A_xB_x + A_yB_y + A_zB_z.
\end{aligned} \qquad [35]$$

The vector product of $\mathbf{A}$ by $\mathbf{B}$ (not $\mathbf{B}$ by $\mathbf{A}$) is a vector defined by the equation

$$\mathbf{A} \times \mathbf{B} = \boldsymbol{\epsilon} AB \sin\theta,$$

where θ is the angle between the positive directions of $\mathbf{A}$ and $\mathbf{B}$ and $\boldsymbol{\epsilon}$ is a unit vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. The positive direction of $\mathbf{A} \times \mathbf{B}$ is defined to be perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ in the sense of advance of a right-handed screw from the first to the second of these vectors through the smaller angle between their posi-

tive directions. (See Fig. 1·19*a*.) Consequently, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Also, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. It is evident that the vector product of **A** and **B** can be considered as a vector with magnitude equal to the area of the parallelogram having **A** and **B** as sides and with the direction of the normal to the plane of **A** and **B**.

It can be proved that the distributive law of multiplication, namely $(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C})$, holds for vector products as well as for scalar products. In view of this and the above relations be-

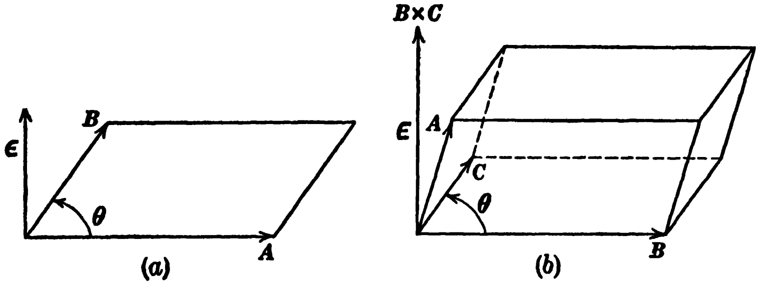

FIG. 1·19

tween **i**, **j**, and **k**, we can express $\mathbf{A} \times \mathbf{B}$ in terms of its **i**, **j**, **k** components as follows:

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \times (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) \\
&= \mathbf{i} \times \mathbf{i}A_xB_x + \mathbf{i} \times \mathbf{j}A_xB_y + \mathbf{i} \times \mathbf{k}A_xB_z \\
&\quad + \mathbf{j} \times \mathbf{i}A_yB_x + \mathbf{j} \times \mathbf{j}A_yB_y + \mathbf{j} \times \mathbf{k}A_yB_z \\
&\quad + \mathbf{k} \times \mathbf{i}A_zB_x + \mathbf{k} \times \mathbf{j}A_zB_y + \mathbf{k} \times \mathbf{k}A_zB_z \\
&= \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) \\
&\quad + \mathbf{k}(A_xB_y - A_yB_x).
\end{aligned}$$

The vector product can be written as the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \qquad [36]$$

(*d*) *Triple scalar product.* The product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is a scalar called the triple scalar product. Inspection of Fig. 1·19*b* shows that this product is equal to the volume of a parallelepiped with edges **A**, **B**, and **C**.

Since interchanging the terms in a scalar product does not change the sign of the product whereas interchanging the terms in a vector product does change the sign of the product, it follows that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = -(\mathbf{C}\times\mathbf{B})\cdot\mathbf{A} = -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}).$$

Since the volume of the parallelepiped remains the same, no matter which face is considered as base, it follows that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}), \text{ etc.} \qquad [37]$$

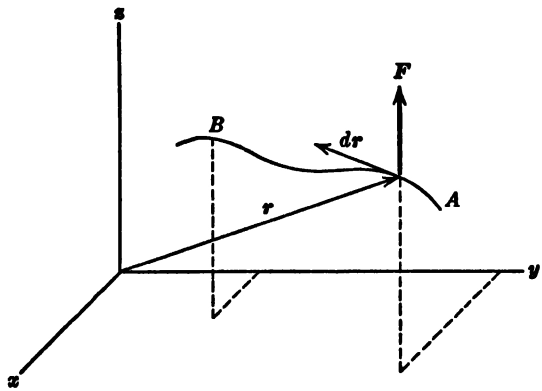

FIG. 1·20

Thus the dot and cross may be interchanged at will and the sign of the product remains unchanged so long as the cyclic order of the vectors remains the same. The triple scalar product can be written as

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \qquad [38]$$

(*e*) *Triple vector product.* The product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is defined as the triple vector product. The vector product of $\mathbf{B}\times\mathbf{C}$ should be formed first, and then the product of $\mathbf{A}$ with this result. The final result is

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \qquad [39]$$

(See example 5, problem set VII.)

(*f*) *Line integrals involving vectors.* The integral $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is a line integral. The vector $d\mathbf{r}$ is taken along the tangent to the curve AB as in Fig. 1·20, and the vector $\mathbf{F}$ may vary in both magnitude and direction along the curve.

Alternative forms are

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B F\cos\theta\, dr,$$

and

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B (\mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z)\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$
$$= \int_A^B (F_x\,dx + F_y\,dy + F_z\,dz).$$

If $\mathbf{F}$ represents a force on a body, then $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is the work done *by* the force as the body moves over the specified path from A to B.

EXAMPLE 1. To fix the ideas more clearly, let F be the force of gravity. Let the curve AB (Fig. 1·21) be one-quarter of the circumference of a circle. Determine the work W done in moving a mass m *against* the force of gravity from A to B along the curve AB in the yz-plane. Consider no friction. Then $\mathbf{F} = -mg\mathbf{k}$. (The minus sign indicates that the force is in the direction of negative $\mathbf{k}$). Then

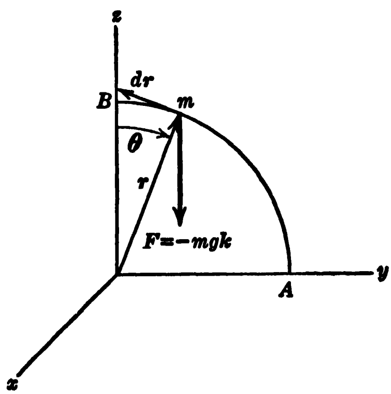

FIG. 1·21

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{j}y + \mathbf{k}z,$$
$$d\mathbf{r} = \mathbf{j}\,dy + \mathbf{k}\,dz,$$
$$z = r\cos\theta,$$
$$dz = -r\sin\theta\,d\theta,$$

and

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_A^B (-mg\,\mathbf{k})\cdot(\mathbf{j}\,dy + \mathbf{k}\,dz)$$
$$= \int_0^{\pi/2} (-mg\,\mathbf{k})\cdot(-r\sin\theta\,d\theta)\mathbf{k}$$
$$= mgr\int_0^{\pi/2} \sin\theta\,d\theta = mgr.$$

This is, of course, the work done in raising the mass m a vertical distance r. If $\mathbf{F}$ varied both in magnitude and direction and AB were a complicated curve, the integrations would be more complicated but no additional principles would be involved.

(*g*) *Derivatives of vectors.* Let $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, where $x = x(t)$, $y = y(t)$, $z = z(t)$, and t is any real parameter, usually the time. If the initial point of $\mathbf{r}$ is fixed at the origin, the terminal point of $\mathbf{r}$ varies and describes a space curve as t varies. Let A and B be two near-by points on this curve. (Fig. 1·22.) Then

$$\begin{aligned}\Delta\mathbf{r} &= \overline{AB} = \mathbf{r}_1 - \mathbf{r} \quad (\mathbf{r}_1 \text{ is not a unit vector.})\\ &= \mathbf{i}x_1 + \mathbf{j}y_1 + \mathbf{k}z_1 - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z\\ &= \mathbf{i}(x_1 - x) + \mathbf{j}(y_1 - y) + \mathbf{k}(z_1 - z)\\ &= \mathbf{i}\Delta x + \mathbf{j}\Delta y + \mathbf{k}\Delta z.\end{aligned}$$

Dividing by Δt and taking the limit as Δt approaches zero, we have

$$\frac{d\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}.$$

It is evident that, as B approaches A (Fig. 1·22), the vector representing $\Delta\mathbf{r}$ approaches the position of the tangent to the curve at A. Hence, $d\mathbf{r}/dt$ is a vector tangent to the space curve described by the terminus of $\mathbf{r}$. Thus, it follows that the derivative of a vector having constant magnitude but variable direction is a vector perpendicular to the differentiated vector.

By procedure similar to the above.

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{i}\frac{d^2x}{dt^2} + \mathbf{j}\frac{d^2y}{dt^2} + \mathbf{k}\frac{d^2z}{dt^2}.$$

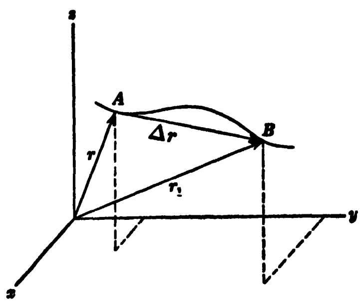

FIG. 1·22

Formulas for differentiating $\mathbf{P}\cdot\mathbf{Q}$ and $\mathbf{P}\times\mathbf{Q}$ can be obtained by expressing each product in its expanded form (Eqs. 35–36) and taking derivatives of these forms. Thus,

$$\begin{aligned}\frac{d}{dt}(\mathbf{P}\cdot\mathbf{Q}) &= \frac{dP_x}{dt}Q_x + \frac{dP_y}{dt}Q_y + \frac{dP_z}{dt}Q_z + P_x\frac{dQ_x}{dt} + P_y\frac{dQ_y}{dt} + P_z\frac{dQ_z}{dt}\\ &= \frac{d\mathbf{P}}{dt}\cdot\mathbf{Q} + \mathbf{P}\cdot\frac{d\mathbf{Q}}{dt},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}(\mathbf{P}\times\mathbf{Q}) &= \mathbf{i}\left[\frac{dP_y}{dt}Q_z - \frac{dP_z}{dt}Q_y + P_y\frac{dQ_z}{dt} - P_z\frac{dQ_y}{dt}\right] + \cdots\\ &= \frac{d\mathbf{P}}{dt}\times\mathbf{Q} + \mathbf{P}\times\frac{d\mathbf{Q}}{dt}.\end{aligned}$$

Both products are differentiated by differentiating the factors just as in the case of scalar products, paying no attention to the dot or cross. It is important to notice, however, that in taking the derivative of the vector product that the order of the vectors must not be changed unless the sign is changed.

EXERCISES VII

1. Compute both the scalar and vector products of the pairs of vectors

$$\begin{cases}\mathbf{A} = 3\mathbf{i} + 0.4\mathbf{j} + 6\mathbf{k}, \\ \mathbf{B} = 0.4\mathbf{i} + 0\mathbf{j} + 8\mathbf{k}.\end{cases}$$

$$\begin{cases}\mathbf{C} = 0.6\mathbf{i} - 7\mathbf{j} - 8\mathbf{k}, \\ \mathbf{D} = \mathbf{i} - \mathbf{j} - \mathbf{k}.\end{cases}$$

2. Find the projection of the vector $\mathbf{A} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ on the line passing through (*a*) the origin and the point $(-2, 3, 7)$, (*b*) the points $(-2, 3, 7)$ and $(1, 2, 3)$.

3. Compute by vector methods the area of the triangle whose vertices are $(3, 4, 2)$, $(1, 0, 5)$, and $(-1, -2, 3)$.

4. If $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\boldsymbol{\theta} = \theta_1\mathbf{i} + \theta_2\mathbf{j} + \theta_3\mathbf{k}$ compute the projection of $\boldsymbol{\theta} \times \mathbf{r}_1$ on the line passing through the points (x_2, y_2, z_2) and (x_3, y_3, z_3).

5. Let $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Perform the expansions $\mathbf{A}\cdot\mathbf{C}$, $\mathbf{A}\cdot\mathbf{B}$, and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, and show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

also

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}.$$

6. Let the curve joining the points A and B in Fig. 1·21 be the hypocycloid $x^{2/3} + y^{2/3} = r^{2/3}$. Compute, by vector methods, the work done against gravity in moving the mass m from A to B along this curve.

7. If the length of the tangent to the curve $x = a \sin t$, $y = b \cos t$, $z = ct^2$ is α at the point $t = \pi/4$ find the projection of this tangent on the line through the origin and the point (a, b, c).

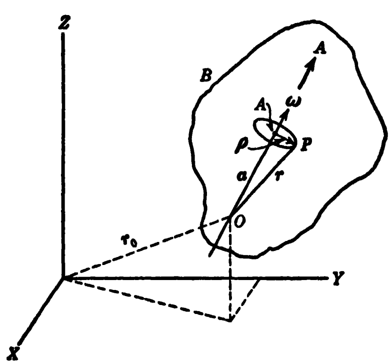

FIG. 1·23

1·17. Kinetic Energy of a Rigid Body. Consider a rigid body B which has angular speed of rotation ω about a point O which in turn has velocity $\mathbf{v}_0$ with respect to the fixed axes X, Y, Z. If the body is rotating about the instantaneous axis OA with angular speed ω then the linear speed of rotation of P is $\rho\omega$. The angular speed becomes a vector if it is assigned a direction. Accordingly, let the vector $\boldsymbol{\omega}$ coincide with OA as shown in Fig. 1·23. The

linear velocity of P with respect to the axis of rotation is $\boldsymbol{\omega} \times \boldsymbol{\rho}$ and the total velocity $\mathbf{V}$ of P with respect to the fixed axes X, Y, Z is

$$\begin{aligned} \mathbf{V} &= \dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \boldsymbol{\rho} \\ &= \dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a}) \qquad [40] \\ &= \dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r}. \end{aligned}$$

Let axes x, y, z fixed in the body (and rotating with the body) be so taken that the origin of coordinates is at the point O which is not necessarily the center of gravity of the body. The vectors $\mathbf{r}$ and $\boldsymbol{\omega}$, expressed in components along the x, y, z axes, are $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ and $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. The kinetic energy T of a particle of mass m_i at P is

$$T = \tfrac{1}{2}m_i V^2 = \tfrac{1}{2}m_i(\dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r}_i)\cdot(\dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r}_i)$$

and the kinetic energy T of the rigid body is

$$T = \tfrac{1}{2}\Sigma m_i(\dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r}_i)\cdot(\dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r}_i), \qquad [41]$$

where $\dot{\mathbf{r}}_0$ is the velocity of the point O with respect to X, Y, Z and the summation is over all particles of the body.

Carrying out the vector operations indicated in Eq. (41) we have

$$T = \tfrac{1}{2}m\dot{r}_0^2 + \Sigma m_i\dot{\mathbf{r}}_0\cdot(\boldsymbol{\omega} \times \mathbf{r}_i) + \tfrac{1}{2}\Sigma m_i(\boldsymbol{\omega} \times \mathbf{r}_i)\cdot(\boldsymbol{\omega} \times \mathbf{r}_i),$$

where m is the mass of B. The last summation in the expression for T is simplified as follows.

$$\begin{aligned} \tfrac{1}{2}\Sigma m_i(\boldsymbol{\omega} \times \mathbf{r}_i)\cdot(\boldsymbol{\omega} \times \mathbf{r}_i) &= \tfrac{1}{2}\Sigma m_i\boldsymbol{\omega}\cdot[\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= \tfrac{1}{2}\Sigma m_i\boldsymbol{\omega}\cdot[\boldsymbol{\omega}(\mathbf{r}_i\cdot\mathbf{r}_i) - \mathbf{r}_i(\mathbf{r}_i\cdot\boldsymbol{\omega})] \quad \text{(see Eq. 39)} \\ &= \tfrac{1}{2}\Sigma m_i\boldsymbol{\omega}\cdot[\boldsymbol{\omega}r_i^2 - \mathbf{r}_i(x_i\omega_x + y_i\omega_y + z_i\omega_z)] \\ &= \tfrac{1}{2}\Sigma m_i[(y_i^2 + z_i^2)\omega_x^2 - x_iy_i\omega_x\omega_y - x_iz_i\omega_x\omega_z \\ &\qquad -y_ix_i\omega_y\omega_x + (z_i^2 + x_i^2)\omega_y^2 - y_iz_i\omega_y\omega_z \\ &\qquad -z_ix_i\omega_z\omega_x - z_iy_i\omega_z\omega_y + (x_i^2 + y_i^2)\omega_z^2] \\ &= \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_x\omega_y - 2E\omega_x\omega_z - 2F\omega_y\omega_z), \end{aligned}$$

where the constants A, B, C, D, E, and F, for a continuous body, are

$$A = \int_v (y^2 + z^2)\sigma\,dv, \quad B = \int_v (x^2 + z^2)\sigma\,dv, \quad C = \int_v (x^2 + y^2)\sigma\,dv$$

$$D = \int_v xy\,\sigma\,dv, \qquad E = \int_v xz\,\sigma\,dv, \qquad F = \int_v yz\,\sigma\,dv,$$

and σ is the volume density.

Finally, the total kinetic energy of a rigid body, when the point O is chosen at random in the body, is

$$T = \tfrac{1}{2}m\dot{r}_0^2 + \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_x\omega_y - 2E\omega_x\omega_z - 2F\omega_y\omega_z) + \Sigma m_i\dot{\mathbf{r}}_o\cdot(\boldsymbol{\omega} \times \mathbf{r}_i). \quad [42]$$

This general expression for T is simplified if the point O is properly located in the body. For example, if (*a*) any point of the body is fixed relative to X, Y, Z and if the origin of the axes x, y, and z is taken at this fixed point the $\dot{\mathbf{r}}_0 = 0$ and

$$T = \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_x\omega_y - 2E\omega_x\omega_z - 2F\omega_y\omega_z), \quad [43]$$

(*b*) no point of the body is fixed relative to X, Y, Z, but the origin O of the axes x, y, and z is taken at the center of gravity of the body then

$$T = \tfrac{1}{2}m\dot{r}_0^2 + \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_x\omega_y - 2E\omega_x\omega_z - 2F\omega_y\omega_z). \quad [44]$$

This value for T is Koenig's theorem.

The constants A, B, C and D, E, F in Eq. (42) are respectively the three principal moments of inertia and the three products of inertia. If the axes x, y, and z are taken along axes of symmetry of the body then $D = E = F = 0$.

1·18. Work Done on a Rigid Body. Let the external force acting on the ith particle of a rigid body be denoted by $\mathbf{F}_i$. Let the displacement relative to fixed axes X, Y, and Z (Fig. 1·23) and due to the force $\mathbf{F}_i$ be $d\mathbf{R}_i$. Then the work done on the body by the forces $\mathbf{F}_i$ $(i = 1, 2, \cdots)$ in the displacements $d\mathbf{R}_i (i = 1, 2, \cdots)$ is

$$\delta W = \Sigma\mathbf{F}_i\cdot d\mathbf{R}_i = \Sigma\mathbf{F}_i\cdot\mathbf{V}_i\,dt, \quad [45]$$

where $\mathbf{V}_i$ is the velocity of the ith particle during the time dt and the summation is taken over all points of the body. (The symbol δW, instead of dW, indicates that the work is not necessarily an exact differential of the coordinates of the system.) In view of Eq. (40) the work is

$$\begin{aligned} \delta W &= \Sigma\mathbf{F}_i\cdot(\dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r}_i)dt \\ &= (\Sigma\mathbf{F}_i)\cdot\dot{\mathbf{r}}_0\,dt + \boldsymbol{\omega}\cdot\Sigma(\mathbf{r}_i \times \mathbf{F}_i)dt \qquad [46] \\ &= \mathbf{F}\cdot\dot{\mathbf{r}}_0\,dt + \mathbf{L}\cdot\boldsymbol{\omega}\,dt, \end{aligned}$$

where $\mathbf{F} = \Sigma\mathbf{F}_i$ is the resultant of the external forces and $\mathbf{L} = \Sigma(\mathbf{r}_i \times \mathbf{F}_i)$ is the resultant external torque acting on the body. The nature of the computation of $\mathbf{F}$ and $\mathbf{L}$ in the general Eq. (46) obviously depends upon the system.

1·19. Potential Energy of Spring-mounted Systems. In certain cases δW is an exact differential in the coordinates of the system or is sufficiently close to an exact differential that a useful approximation which is an exact differential can be obtained. Such an approximation to δW is obtainable in the important engineering case of spring-mounted systems.

In Fig. 1·24 let X_0, Y_0, Z_0 be the coordinates of the center of gravity of the cylinder (or any rigid body). Take axes x, y, and z fixed

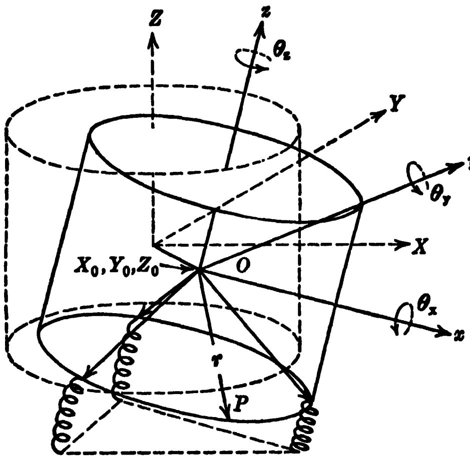

FIG. 1·24. Mounting of Refrigerator Unit.

in the body and with origin O at the center of gravity. Let $\mathbf{r}$ be the vector from O to a general point P of the body. It is desired to obtain a formula for the general displacement $\mathbf{S}$ of P during a small translation and rotation of the body such as occurs in vibratory motion. Let the unit vectors along x, y, z be denoted by $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$.

Suppose the body is in equilibrium (dotted) configuration at $t = 0$. At time t the displacement of P is

$$\mathbf{S} = \int_0^t \dot{\mathbf{V}}\,dt = \int_0^t (\dot{\mathbf{r}}_0 + \boldsymbol{\omega} \times \mathbf{r})dt \qquad \text{(see Eq. 40)}$$

$$= \mathbf{S}_0 + \int_0^t [(\omega_y r_z - \omega_z r_y)\mathbf{I} + (\omega_z r_x - \omega_x r_z)\mathbf{J} + (\omega_x r_y - \omega_y r_x)\mathbf{K}]dt,$$

where $\mathbf{S}_0$ is the displacement of the center of gravity from its equilibrium position. Now r_x, r_y, r_z are the projections on the x, y, z axes and consequently are constants. The unit vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are along the axes x, y, and z and they do change in direction with the time. Sup-

pose these unit vectors are assumed constant by replacing them by the unit vectors **i**, **j**, **k** along the axes X, Y, and Z. The error made in this assumption involves the cosines of the angles between **I** and **i**, **J** and **j**, etc., and in vibratory motion these cosines are approximately unity since the changes in the cosines of the small angles are much smaller than the variation in the small angles themselves. The total displacement of P at any time is

$$\mathbf{S} = \mathbf{S}_0 + (\theta_y r_z - \theta_z r_y)\mathbf{i} + (\theta_z r_x - \theta_x r_z)\mathbf{j} + (\theta_x r_y - \theta_y r_x)\mathbf{k}$$

$$= \mathbf{S}_0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \theta_x & \theta_y & \theta_z \\ r_x & r_y & r_z \end{vmatrix}, \qquad [47]$$

where the positive directions of θ_x, θ_y, and θ_z are indicated in the figure. Evidently the X, Y, and Z components of **S** are respectively $\mathbf{i}\cdot\mathbf{S}$, $\mathbf{j}\cdot\mathbf{S}$, $\mathbf{k}\cdot\mathbf{S}$ or

$$X = X_0 + (\theta_y r_z - \theta_z r_y),$$
$$Y = Y_0 + (\theta_z r_x - \theta_x r_z),$$
$$Z = Z_0 + (\theta_x r_y - \theta_y r_x).$$

Suppose that Hooke's law holds in compressing a spring; then the force in a compression in the z direction is

$$\mathbf{F} = \lambda[Z_0 + (\theta_x r_y - \theta_y r_x)]\mathbf{k}$$

and the work done (Eq. 45) against the spring at P is

$$W_{P_z} = \int_0^t \mathbf{F}\cdot\mathbf{V}\,dt = \int_0^t \lambda[Z_0 + (\theta_x r_y - \theta_y r_x)]\frac{d}{dt}[Z_0 + (\theta_x r_y - \theta_y r_x)]dt$$

$$= \frac{\lambda}{2}[Z_0 + (\theta_x r_y - \theta_y r_x)]^2, \qquad [48]$$

where λ is the spring constant. The total potential energy is the work done against all the springs in a displacement from equilibrium position.

1·20. Differential Equations of Oscillations of Spring-mounted Motor. A motor of mass m is mounted on four identical springs as shown in Fig. 1·25. The spring constants of a single spring in a vertical displacement is k and in any horizontal displacement is k_0. The distances between the springs S_1S_2 and S_2S_3 are respectively $2a$ and $2b$. The center of gravity of the motor is located at its geometrical center and at a distance R above the upper end of the springs. The moments of inertia of the system about the shaft of the motor and about any

line perpendicular to the shaft and through the center of gravity are respectively I and I_0. Under the assumption that the gyroscopic effect of the rotor on small oscillations of the motor is negligible, write the differential equations of these small oscillations.

Take the origin of the axes x, y, and z at the center of gravity of the motor and their direction at time t as shown in the figure. Denote the coordinates of the center of gravity relative to the fixed axes X, Y, and Z by X_0, Y_0, Z_0. Let the generalized coordinates defining the position of the motor at any time be $X_0, Y_0, Z_0, \xi = \theta_x, \eta = \theta_y$, and $\zeta = \theta_z$

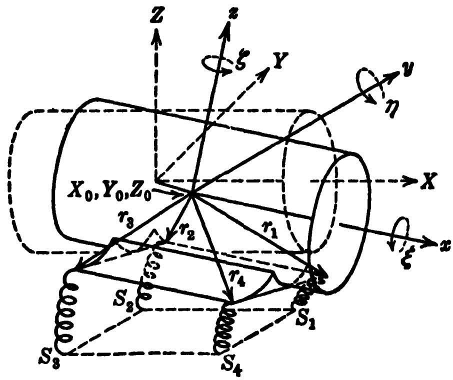

FIG. 1·25. Spring-mounting of Electric Motor.

and let the positive directions of the small rotations be as indicated in the figure.

From the figure the values of the position vectors are

$$\begin{aligned}
\mathbf{r}_1 &= a\mathbf{I} + b\mathbf{J} - R\mathbf{K} \doteq a\mathbf{i} + b\mathbf{j} - R\mathbf{k},\\
\mathbf{r}_2 &= -a\mathbf{I} + b\mathbf{J} - R\mathbf{K} \doteq -a\mathbf{i} + b\mathbf{j} - R\mathbf{k},\\
\mathbf{r}_3 &= -a\mathbf{I} - b\mathbf{J} - R\mathbf{K} \doteq -a\mathbf{i} - b\mathbf{j} - R\mathbf{k},\\
\mathbf{r}_4 &= a\mathbf{I} - b\mathbf{J} - R\mathbf{K} \doteq a\mathbf{i} - b\mathbf{j} - R\mathbf{k}.
\end{aligned}$$

By Eq. (47) the vertical displacements of the upper ends of the springs are

$$S_{1z} = \mathbf{k}\cdot\mathbf{S} = \mathbf{k}\cdot\mathbf{S}_0 + \mathbf{k}\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \xi & \eta & \zeta \\ a & b & -R \end{vmatrix} = Z_0 + (b\xi - a\eta),$$

$$\begin{aligned}
S_{2z} &= Z_0 + (b\xi + a\eta),\\
S_{3z} &= Z_0 + (-b\xi + a\eta),\\
S_{4z} &= Z_0 + (-b\xi - a\eta).
\end{aligned}$$

Similarly, the two X and Y horizontal displacements of the tops of the springs are

$$\begin{aligned}
S_{1x} &= X_0 + (-R\eta - b\zeta), & S_{1y} &= Y_0 + (R\xi + a\zeta),\\
S_{2x} &= X_0 + (-R\eta - b\zeta), & S_{2y} &= Y_0 + (R\xi - a\zeta),\\
S_{3x} &= X_0 + (-R\eta + b\zeta), & S_{3y} &= Y_0 + (R\xi - a\zeta),\\
S_{4x} &= X_0 + (-R\eta + b\zeta), & S_{4y} &= Y_0 + (R\xi + a\zeta).
\end{aligned}$$

By Eq. (48) the potential energy due to the vertical compression of S_{1z} is

$$\frac{k}{2}(Z_0 + b\xi - a\eta)^2$$

and the total potential energy of all four springs, or of the system, is

$$\begin{aligned}
V = \frac{k}{2}&[(Z_0 + b\xi - a\eta)^2 + (Z_0 + b\xi + a\eta)^2 + (Z_0 - b\xi + a\eta)^2\\
&\quad + (Z_0 - b\xi - a\eta)^2]\\
&+\frac{k_0}{2}[(X_0 - R\eta - b\zeta)^2 + (X_0 - R\eta - b\zeta)^2 + (X_0 - R\eta + b\zeta)^2\\
&\quad + (X_0 - R\eta + b\zeta)^2]\\
&+\frac{k_0}{2}[(Y_0 + R\xi + a\zeta)^2 + (Y_0 + R\xi - a\zeta)^2 + (Y_0 + R\xi - a\zeta)^2\\
&\quad + (Y_0 + R\xi + a\zeta)^2].
\end{aligned}$$

The kinetic energy by Eq. (44) is

$$T = \tfrac{1}{2}[M(\dot{X}_0^2 + \dot{Y}_0^2 + \dot{Z}_0^2) + (I\dot{\xi}^2 + I_0\dot{\eta}^2 + I_0\dot{\zeta}^2)].$$

The axes in the motor are taken so that the products of inertia vanish.

Equations (31), where $Q_r = -\dfrac{\partial V}{\partial q_r}$, $q_1 = X_0$, $q_2 = Y_0$, $q_3 = Z_0$, $q_4 = \xi$, $q_5 = \eta$, $q_6 = \zeta$ yield the differential equations of motion

$$\begin{aligned}
&M\ddot{X}_0 + 4k_0(X_0 - R\eta) = 0,\\
&M\ddot{Y}_0 + 4k_0(Y_0 + R\xi) = 0,\\
&M\ddot{Z}_0 + 4kZ_0 = 0,\\
&I\ddot{\xi} \quad + 4(kb^2 + k_0R^2)\xi + 4k_0RY_0 = 0,\\
&I_0\ddot{\eta} \quad + 4(ka^2 + k_0R^2)\eta - 4k_0RX_0 = 0,\\
&I_0\ddot{\zeta} \quad + 4k_0(a^2 + b^2)\zeta = 0.
\end{aligned} \qquad [49]$$

Obviously, not all of these equations are independent. Suitable methods of solving equations of this type are given in Sec. 4 (Theory of Vibrations) and Sec. 7 (Rayleigh's principle) of this chapter.

1·21. Potential Energy of Electric Locomotives. The forces acting on an electric locomotive fall into four groups (*a*) spring forces, (*b*) creepage forces, (*c*) flange forces, and (*d*) damping forces. The first

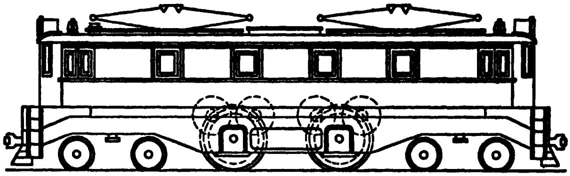

FIG. 1·26. Electric Locomotive, Type 2—B±2.

set is easily found by the method of §1·19. Figure 1·26 shows the spring arrangement of one of the mechanically simplest types of high-speed electric locomotives, denoted as type 2—B ± 2. One guiding truck is independent of the driving truck while the other is articulated with the driving truck in such a way that the entire spring system (four nests of springs) is equivalent to three-point support on three

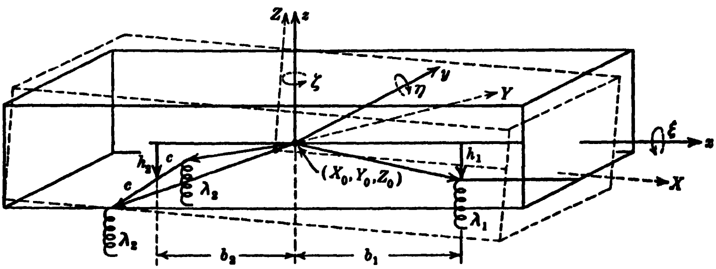

FIG. 1·27. Schematic Three-point Support of Electric Locomotive.

springs as indicated in Fig. 1.27. The journal construction is such that the springs are subject to compression only. The locomotive is equipped with a quill drive so that practically all the weight of the locomotive is spring-mounted. The mass of the locomotive is M and its dimensions are shown in Fig. 1·27. Let it be required to find the potential energy stored in the springs.

Let the height of the center of gravity of the spring-borne mass above the upper ends of the front and rear springs in equilibrium be respectively, h_1 and h_2. The distance measured, parallel to the track from the center of gravity to a point above the front spring is b_1 and from the center of gravity to a point above and midway between the

rear springs is b_2. The distance between the two rear springs is $2c$. Let the origin of the coordinate system XYZ be taken (Fig. 1·27) at the center of gravity of the spring-mounted mass. Let the angular displacements about the x, y, and z axes be respectively ξ, η, ζ and the spring constant of the front spring be λ_1 and of each of the two rear springs be λ_2. By Eq. (47) the vertical components of the displacements of the upper ends of the three equivalent springs are

$$\begin{aligned} S_{1z} &= \mathbf{k}\cdot\mathbf{S} + \mathbf{k}\cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \xi & \eta & \zeta \\ b_1 & 0 & -h_1 \end{vmatrix}, \\ &= Z_0 - b_1\eta, \\ S_{2z} &= Z_0 + c\xi + b_2\eta, \\ S_{3z} &= Z_0 - c\xi + b_2\eta. \end{aligned}$$

By Eq. (48) the total potential energy V of the cab is

$$V = \frac{\lambda_1}{2}(Z_0 - b_1\eta)^2 + \frac{\lambda_2}{2}(Z_0 + c\xi + b_2\eta)^2 + \frac{\lambda_2}{2}(Z_0 - c\xi + b_2\eta)^2.$$

The differential equations of motion for a locomotive are set up in § 1·32.

1·22. Differential Equations of Motion of a Gyroscope. The motion of a gyroscope is an example of the motion of a rigid body about a point which is both fixed in the body and in space. Let the origin O of the system of axes X, Y, and Z fixed in space be taken at the fixed point of the rigid body. Let the origin of the axes x, y, and z fixed in the body but moving with respect to XYZ be taken also at the fixed point. If the position of the axes x, y, and z with respect to XYZ can be found at time t then obviously the position of the rigid body is known. A coordinate system is desired which relates, in a simple way, a general position of the moving axes x, y, and z to the fixed axes X, Y, and Z. One such system is Euler's angles (Fig. 1·28). A selected point of the rigid body can be brought from any initial to a given final position by means of three angular displacements. To fix the ideas suppose that the two sets of axes initially coincide. Beginning with the axes coincident (a) rotate the axis ox through the angle ψ to the position OX_1; (b) next, rotate the x, y, and z axes (and the rigid body) through the angle θ about the line OX_1 or Ox; (c) finally, rotate the x, y, and z axes through the angle φ about the line (axis) Oz. These

three angular displacements give the final position of the axes x, y, and z with respect to the axes X, Y, and Z.

In computing by means of Eq. (43) the kinetic energy of the body, it is necessary to know the projections ω_x, ω_y, and ω_z of the angular velocity about the moving axes x, y, and z as functions of ψ, θ, φ and their derivatives with respect to the time. To obtain these projections, first resolve the vector ω onto the lines OZ, OX_1, and Oz. The vector sum of these projections is, of course, ω. These components are

$\dot{\psi}$ about OZ, $\dot{\theta}$ about OX_1, and $\dot{\varphi}$ about Oz,

where the dots indicate derivatives with respect to the time.

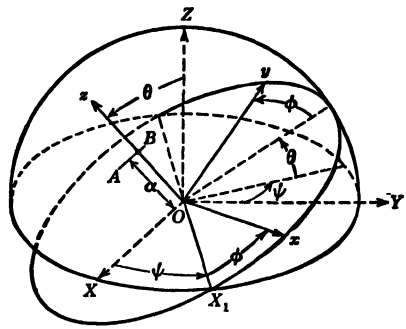

FIG. 1·28. Euler's Angles and Gyroscope.

From Fig. 1·28 the projections of these angular velocities onto the x, y, and z moving axes are easily seen to be

$$\begin{aligned} \omega_x &= \dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi, \\ \omega_y &= \dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi, \\ \omega_z &= \dot{\psi} \cos\theta + \dot{\varphi}. \end{aligned} \tag{50}$$

The kinetic energy of a rigid body rotating about a point of the body fixed with respect to the axes X, Y, and Z and having moments and products of inertia A, B, C; D, E, and F is given by the substitution of (50) in (43).

The derivation of the differential equations of motion of a gyroscope is now merely routine computation. The instantaneous angular velocity ω does not necessarily coincide with the axis of spin of the gyroscope. In fact ω may be entirely outside the rotating body. Let Oz be the axis of spin of the gyroscope. Let the x and y axes be taken parallel to the plane AB of the gyroscope (Fig. 1·28) and let the moments of inertia about these axes be A, and about the z axis be C. Then by Eqs. (43) and (50) the kinetic energy T is

$$\begin{aligned} T &= \tfrac{1}{2}(A\omega_x^2 + A\omega_y^2 + C\omega_z^2) \\ &= \tfrac{1}{2}[A(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + C(\dot{\varphi} + \dot{\psi}\cos\theta)^2]. \end{aligned}$$

The only external force acting on the gyroscope is the force of gravity mg acting at the center of gravity (Fig. 1·28).

Since

$$\frac{\partial T}{\partial \dot{\psi}} = A\dot{\psi} \sin^2 \theta + (C \cos \theta)(\dot{\varphi} + \dot{\psi} \cos \theta),$$

$$\frac{\partial T}{\partial \dot{\theta}} = A\dot{\theta},$$

$$\frac{\partial T}{\partial \dot{\varphi}} = C(\dot{\varphi} + \dot{\psi} \cos \theta),$$

$$\frac{\partial T}{\partial \psi} = 0,$$

$$\frac{\partial T}{\partial \theta} = A\dot{\psi}^2 \sin \theta \cos \theta - C(\dot{\varphi} + \dot{\psi} \cos \theta)\dot{\psi} \sin \theta,$$

$$\frac{\partial T}{\partial \varphi} = 0,$$

and the torque Q_θ about OX_1 is $Q_\theta = mg\alpha \sin \theta$, it follows that Eqs. (31) yield by routine substitutions the following differential equations of motion of the gyroscope

$$\frac{d}{dt} [A\dot{\psi} \sin^2 \theta + (C \cos \theta)(\dot{\psi} \cos \theta + \dot{\varphi})] = 0,$$

$$A\ddot{\theta} - A\dot{\psi}^2 \sin \theta \cos \theta + C(\dot{\psi} \cos \theta + \dot{\varphi})\dot{\psi} \sin \theta = mg\alpha \sin \theta, \qquad [51]$$

$$C \frac{d}{dt} [\dot{\psi} \cos \theta + \dot{\varphi}] = 0.$$

1·23. Euler's Equations for a Rigid Body Containing a Fixed Point. Euler developed three important differential equations of the motion of a rigid body containing a fixed point. These equations give for every instant the time variation of the angular velocity components ω_x, ω_y, ω_z, about the principal axes of the body (axes x, y, and z, §1·22) in terms of the external moments L, M, N acting respectively about the axes x, y, and z and products of the same velocity components. Consequently, if the moments about the principal axes of a rigid body are known, then ω_x, ω_y, and ω_z can be expressed in terms of these moments. Thus $\boldsymbol{\omega}$, the instantaneous angular velocity at any time, can be found. Conversely, if $\boldsymbol{\omega}$ is known then the moments L, M, N can be found.

The kinetic energy T of the body by Eqs. (43) and (50) is

$$T = \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2)$$
$$= \tfrac{1}{2}[A(\dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi)^2 + B(\dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi)^2 + C(\dot\psi \cos\theta + \dot\varphi)^2],$$

where A, B, and C are the three principal moments of inertia. Now

$$\frac{\partial T}{\partial \dot\varphi} = C(\dot\varphi + \dot\psi \cos\theta) = C\omega_z,$$

$$\frac{\partial T}{\partial \varphi} = A\omega_y(\dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi) - B\omega_x(\dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi)$$
$$= A\omega_x\omega_y - B\omega_y\omega_x.$$

Substituting in Eqs. (31) we obtain the Euler equation

$$C\frac{d\omega_z}{dt} - (A - B)\omega_x\omega_y = N.$$

The remaining two equations are obtained in a similar manner. Thus the three Euler equations of the motion of a rigid body containing a fixed point are

$$A\frac{d\omega_x}{dt} + (C - B)\omega_y\omega_z = L,$$
$$B\frac{d\omega_y}{dt} + (A - C)\omega_z\omega_x = M, \qquad [52]$$
$$C\frac{d\omega_z}{dt} + (B - A)\omega_x\omega_y = N.$$

1·24. Summary of Section 3. The systems under consideration consist of single particles or of a rigid body. Once the kinetic energy, potential energy, if it exists, and the external generalized forces have been computed the derivation of the differential equations of motions by means of formulas (31) is merely a routine matter. The following summary relates to the computation of the above three quantities.

(*a*) *Single particle.* If there are no constraints a single particle possesses three degrees of freedom and consequently requires three coordinates to define its position. The rectangular coordinates of the position of the particle are related to other coordinates (spherical, cylindrical, toroidal, etc.) by means of three equations such as those of example

2, § 1·13. The kinetic energy is given by $T = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. If the particle has gravitational potential energy this energy is equal to the work done in moving the particle against gravity from some reference position (equilibrium position if it exists) to its current or general position. In finding expressions for the generalized forces $Q_r (r = 1, 2 \cdots, n)$ care must be taken to make sure that the product of Q_r by the corresponding displacement δq_r is work done on the system.

If constraints exist then the particle possesses less than three degrees of freedom. The rectangular coordinates x, y, z then are expressible in terms of less than three generalized coordinates such as in the equation of example 1, § 1·13. The statements regarding the energies and generalized forces made for three degrees of freedom hold also for one or two degrees of freedom.

(*b*) *Rigid body.* If the constraints are such that the body has few degrees of freedom such as in example 3, § 1·13 the kinetic energy can be obtained by the application of Koenig's theorem § 1·13. The gravitational potential energy of a rigid body is the work done against gravity in bringing the body from a reference position to a current position.

If the body has six degrees of freedom, then the kinetic energy is given by either Eqs. (42) or (44). The latter is preferable. If x, y, and z are taken along axes of symmetry in the body, then $D = E = F = 0$. If the body possesses potential energy V and if the only external forces acting on the body are $-\dfrac{\partial V}{\partial q_r}$, $(r = 1, 2, \cdots, n)$ then $\delta W = dV$ in (46) and the potential energy is found at once. In the important engineering case of spring-mounted bodies the potential energy is obtained with sufficient accuracy by the routine method of Eqs. (47–48). In the use of this method one vector is drawn from the center of gravity of the rigid body to the upper end of each spring. If external forces, other than the forces of gravity and of springs, act on the system these forces are $Q_r^{(0)}$ of § 1·12. If dissipation forces, proportional to the velocities of their points of application, act on the body then Lagrange's equations for the body are Eqs. (32).

If the body contains one fixed point then its kinetic energy is given by the substitution of (50) in (43). The external torques are taken about the lines OZ, OX_1, and Oz. The position of the body is then given by the solution of the Lagrangian equations for the angles ψ, θ, and φ as functions of the time. An alternative method of studying the motion is by means of Euler's equations. The solution of (52) for the components ω_x, ω_y, and ω_z give the direction of the instantaneous axis of rotation and the magnitude of the instantaneous velocity as func-

tions of the time. When these components are substituted in Eqs. (50) the angular position is given by the solution of the resulting system of differential equations. The torques L, M, and N in Euler's equation are taken about the moving axes x, y, and z.

EXERCISES VIII

1. Two particles m_1 and m_2, connected by a rod of negligible weight, move on a smooth vertical circle. Find the differential equation of motion.

2. A triangular lamina ABC of sides a, b, c is suspended by the vertex A. The lamina swings in its own plane under the influence of gravity. Find the length of the equivalent simple pendulum.

3. A rough uniform circular cylinder of radius r and moment of inertia I has coiled around its middle section a flexible inextensible string. The string is rolled up until the cylinder in a horizontal position touches a fixed point P to which the string is attached. The cylinder is made to revolve in a horizontal plane with angular velocity ω and then released. Find the differential equations of motion.

4. Two masses m_1 and m_2 are connected and suspended by inextensible strings of lengths a and b as shown in Fig 1·29. The masses m_1 and m_2 are pulled aside in opposite directions from the plane $ABCD$ and released. Write the differential equations of motion.

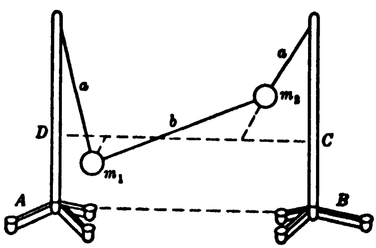

FIG. 1·29

5. A heavy uniform rod is mounted in a frame such that one end of the rod is constrained to move without friction in a horizontal plane, the other end without friction in a vertical groove of the frame. The frame is rotating with constant angular velocity about the vertical groove as an axis. Write the differential equation of motion of the rod.

6. The foot of a ladder is resting on a smooth horizontal plane and its top leans against a lamp post. The top of the ladder slides down the post while the foot of the ladder is free to move in any horizontal direction. Write the differential equations of motion of the ladder.

7. Show that the expression $\frac{1}{2}\Sigma m_i\,(\boldsymbol{\omega} \times \mathbf{r}_i)\cdot(\boldsymbol{\omega} \times \mathbf{r}_i)$, obtained in the reduction of Eq. (41), can be written $\boldsymbol{\omega}\cdot\boldsymbol{\Phi}\cdot\boldsymbol{\omega}/2$, where $\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$ and

$$\begin{aligned}\boldsymbol{\Phi} = \mathbf{ii}\,A &- \mathbf{ij}\,D - \mathbf{ik}\,E\\ - \mathbf{ji}\,D &+ \mathbf{jj}\,B - \mathbf{jk}\,F\\ - \mathbf{ki}\,E &- \mathbf{kj}\,F + \mathbf{kk}\,C,\end{aligned}$$

where $\mathbf{i}\cdot(\mathbf{ii}) = \mathbf{i}\cdot\mathbf{i}(\mathbf{i}) = \mathbf{i}$, $\mathbf{i}\cdot(\mathbf{ji}) = \mathbf{i}\cdot\mathbf{j}(\mathbf{i}) = 0$, etc. The quantity $\boldsymbol{\Phi}$ is known as a dyadic in nonion form.

8. Two masses m_1 and m_2 ($m_2 > m_1$) are suspended from a wheel and axle of radii r_1 and r_2 ($r_2 < r_1$). The moment of inertia of the combined wheel and axle is I. Find the acceleration of m_2.

PROBLEMS IX

1. A mass m_1 is supported by a wheel and elastic tire, and a mass m_2 is supported above m_1 by a spring. (Fig. 1·30.) Constraints permit vertical motion only and the

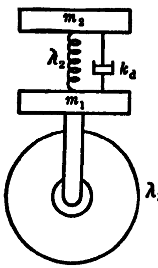

FIG. 1·30. Spring, Tire, and Shock-absorber.

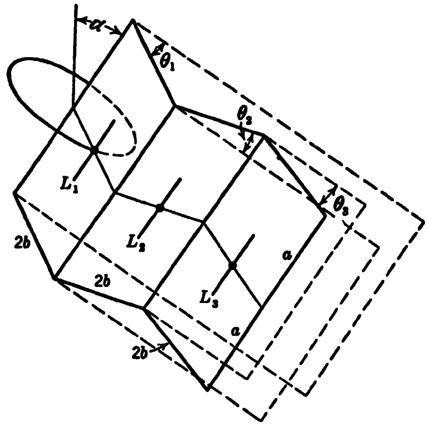

FIG. 1·31. Compound Seismograph.

wheel is not allowed to rotate. A shock-absorber, which acts equally for either direction of motion of its piston is placed in parallel with the spring. The force exerted by

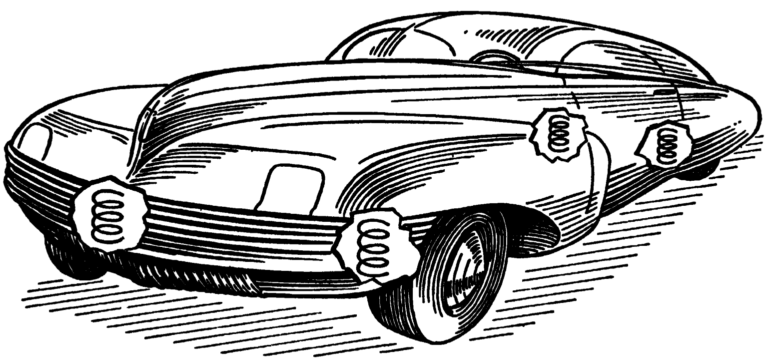

FIG. 1·32

the shock-absorber is always proportional to the difference of the velocities of m_1 and m_2. The system is set in motion. Write the differential equations of motion of m_1 and m_2.

2. Obtain the differential equations of motion of the seismograph shown in Fig. 1·31. The dimensions, inclination, and masses are shown in the figure. Assume there is no damping.

3. Solve problem 2 with the additional conditions that there is relative damping in the system such that the motion of the second plate is damped relative to the first, and the third (lowest) plate is damped relative to the second. Let the damping be proportional to the differences of the first powers of the velocity.

4. Given that the angular displacements from equilibrium position of a spring-mounted mass do not exceed 5°, show by examination of the integral leading to Eq. (47) that the maximum error in the potential energy as given by use of Eq. (47) is less than 2 per cent.

5. The four coiled springs of an automobile are alike in pairs and all obey Hooke's law. Let the spring constant of a rear spring be λ_2 and that of a front spring be λ_1. The distances, measured parallel to the length of the car, from the center of gravity of the car to a point above and midway between the front and rear spring supports are respectively b_1 and b_2. The lateral distances between the springs is c. The height of the center of gravity above the tops of all four springs is h. Compute the potential energy stored in the springs under the assumption that the angular motions are small, 10°.

6. Suppose the forward component of the velocity of the center of gravity of the car in problem 5 is V, a constant. Let the car travel over an undulatory road and each undulation be a sine wave of length L and amplitude y_0. Let the principal moments of inertia about axes through the center of gravity be A, B, and C. Neglecting the effect of the tires and assuming the angular displacements small, write the differential equations of motion of the car.

7. Electric locomotives of the type 2 — C±C — 2 possess six driving axles and two guiding trucks. Each half of the spring-borne weight of the locomotive rests on three driving axles and on one guiding truck by means of three-point support as shown in Fig. 1·33. The locomotive cab rests on two king-pins shown. Very slight lateral rolling motion of the cab is possible before the springs are appreciably acted upon. Let the equivalent spring constants of each of the guiding-truck springs be λ_1 and the spring constants of each of the other equivalent springs be λ_2. Neglecting the small lateral rolling motion described above, compute the potential energy of the spring-borne mass

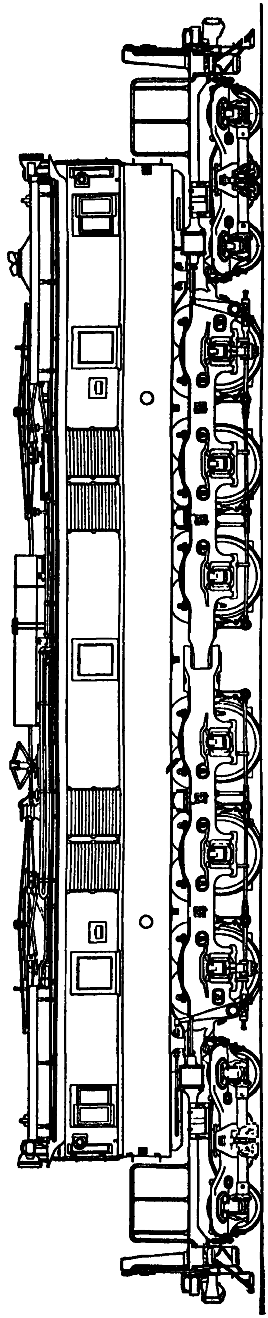

FIG. 1·33. Electric Locomotive, Type 2 — C±C — 2.

of the locomotive. No angular displacement exceeds 3°. (The differential equations of motion are required in problem set XII.)

8. Solve Ex. 4 of set VIII when the inextensible strings have been replaced by elastic bands which obey Hooke's law. Let the modulus of the elastic bands be λ.

9. Suppose the motor of Fig. 1·25 to be mounted on three springs, the two rear springs as shown, and a third front spring under the shaft of the motor. Obtain the differential equations of motion.

10. Obtain from Eqs. (51) the single equation in θ

$$A\ddot{\theta} - \frac{(L - C\alpha\cos\theta)(L\cos\theta - C\alpha)}{A\sin^3\theta} - mg\alpha\sin\theta = 0,$$

where α and L are constants of integration.

(4)

Lagrange's Equations and the Theory of Vibrations

(Normal Coordinates)

Lagrange's equations are of use in writing the differential equations of motion of small oscillations or vibrations of a rigid body about either equilibrium position or about steady motion. Motion about equilibrium configuration is the more important in engineering applications.

1·25. Potential and Kinetic Energies of Oscillating Systems. Let $\theta_1, \theta_2, \cdots, \theta_n$ be the n generalized coordinates of a holonomic dynamical system. Let $\theta_1^{(0)}, \theta_2^{(0)}, \cdots, \theta_n^{(0)}$ be the values of $\theta_1, \theta_2, \cdots, \theta_n$ when the system is in equilibrium position. Make the change of variables of position

$$\theta_i = \theta_i^{(0)} + q_i \quad (i = 1, 2, \cdots, n),$$

where now all q_i vanish in equilibrium position. Denote by V_0 the potential energy of the system in equilibrium configuration. Then the potential energy V in a general position can be written, by aid of Taylor's theorem, as

$$V = V_0 + q_1\frac{\partial V}{\partial\theta_1} + q_2\frac{\partial V}{\partial\theta_2} + \cdots + q_n\frac{\partial V}{\partial\theta_n}$$
$$+ \tfrac{1}{2}\left(q_1^2\frac{\partial^2 V}{\partial\theta_1^2} + 2q_1q_2\frac{\partial^2 V}{\partial\theta_2\partial\theta_1} + \cdots + q_n^2\frac{\partial^2 V}{\partial\theta_n^2}\right) + \cdots \quad [53]$$

where the coefficients $\dfrac{\partial V}{\partial\theta_1}, \dfrac{\partial^2 V}{\partial\theta_1^2}, \dfrac{\partial^2 V}{\partial\theta_2\partial\theta_1}, \cdots$ are evaluated at $\theta_i = \theta_i^{(0)}$ that is, in equilibrium position. The forces acting on a system in equilibrium position are zero. From § 1·8 the forces acting in the directions

of possible displacements of the system are $-\dfrac{\partial V}{\partial \theta_i}$ $(i = 1, 2, \cdots, n)$.

Consequently,

$$\frac{\partial V}{\partial \theta_1} = \frac{\partial V}{\partial \theta_2} = \cdots = \frac{\partial V}{\partial \theta_n} = 0.$$

If the zero of potential energy is taken at equilibrium position, then $V_0 = 0$ and if all motions are small (vibrations or small oscillations) then terms in powers of the q's higher than the second can be omitted and Eq. (53) becomes

$$V = \tfrac{1}{2}(b_{11}q_1^2 + 2b_{12}q_1q_2 + \cdots + b_{nn}q_n^2), \tag{54}$$

where $b_{ij}(i, j = 1, 2, \cdots, n)$ are constants.

Suppose Eqs. (29) do not contain the time explicitly. Then remembering that a general position of the system is denoted by $\theta_1, \theta_2, \cdots, \theta_n$, the kinetic energy T, by the reasoning following Eqs. (29), is

$$\begin{aligned} T = \tfrac{1}{2}\sum m_i\Bigg[&\left(\frac{\partial f_i}{\partial \theta_1}\dot\theta_1 + \cdots + \frac{\partial f_i}{\partial \theta_n}\dot\theta_n\right)^2 \\ &+ \left(\frac{\partial g_i}{\partial \theta_1}\dot\theta_1 + \cdots + \frac{\partial g_i}{\partial \theta_n}\dot\theta_n\right)^2 \\ &+ \left(\frac{\partial h_i}{\partial \theta_1}\dot\theta_1 + \cdots + \frac{\partial h_i}{\partial \theta_n}\dot\theta_n\right)^2\Bigg] \\ = \tfrac{1}{2}\sum m_i\Bigg[&\left(\frac{\partial f_i}{\partial q_1}\dot q_1 + \cdots + \frac{\partial f_i}{\partial q_n}\dot q_n\right)^2 \\ &+ \left(\frac{\partial g_i}{\partial q_1}\dot q_1 + \cdots + \frac{\partial g_i}{\partial q_n}\dot q_n\right)^2 \\ &+ \left(\frac{\partial h_i}{\partial q_1}\dot q_1 + \cdots + \frac{\partial h_i}{\partial q_n}\dot q_n\right)^2\Bigg]. \end{aligned}$$

In general, the coefficients of $\dot q_1, \dot q_2, \cdots, \dot q_n$ are functions of $q_1, q_2, \cdots, q_n$ but since the motions are small we may regard their values at $q_1 = q_2 = \cdots = q_n = 0$ as being their values at any time. Consequently,

$$T = \tfrac{1}{2}(a_{11}\dot q_1^2 + 2a_{12}\dot q_1\dot q_2 + \cdots + a_{nn}\dot q_n^2), \tag{55}$$

where the a_{ij} are constants.

If no forces act on the system other than $-\dfrac{\partial V}{\partial q_i}$, $(i = 1, 2, \cdots, n)$, then Lagrange's equations are obtained by the substitution of Eqs.

(54–55) in Eqs. (31). Equations (49) are an example of the systems in question.

1·26. Solution of Differential Equations of Vibrations with Damping. The method of solution of a system of homogeneous linear differential equations with constant coefficients is made clear by the solution of three equations in three unknowns. (A review of determinants and simultaneous linear homogeneous algebraic equations may be advisable.)[8] Let the equations be

$$\begin{aligned} z_{11}(p)q_1 + z_{12}(p)q_2 + z_{13}(p)q_3 &= 0, \\ z_{21}(p)q_1 + z_{22}(p)q_2 + z_{23}(p)q_3 &= 0, \\ z_{31}(p)q_1 + z_{32}(p)q_2 + z_{33}(p)q_3 &= 0, \end{aligned} \qquad [56]$$

where

$$z_{ij}(p) = a_{ij}p^2 + d_{ij}p + b_{ij}, \quad \text{and} \quad p = \frac{d}{dt}, \quad p^2 = \frac{d^2}{dt^2}.$$

The substitution of

$$q_1 = C_1e^{mt}, \quad q_2 = C_1'e^{mt}, \quad q_3 = C_1''e^{mt} \qquad [57]$$

in Eqs. (56) and the division of each of the resulting equations by e^{mt} yield

$$\begin{aligned} z_{11}(m)C_1 + z_{12}(m)C_1' + z_{13}(m)C_1'' &= 0, \\ z_{21}(m)C_1 + z_{22}(m)C_1' + z_{23}(m)C_1'' &= 0, \\ z_{31}(m)C_1 + z_{32}(m)C_1' + z_{33}(m)C_1'' &= 0. \end{aligned} \qquad [58]$$

In order that Eqs. (58) have a solution in C_1, C_1', C_1'' other than the trivial solution $C_1 = C_1' = C_1'' = 0$, it is necessary and sufficient that the determinant

$$\Delta = \begin{vmatrix} z_{11}(m) & z_{12}(m) & z_{13}(m) \\ z_{21}(m) & z_{22}(m) & z_{23}(m) \\ z_{31}(m) & z_{32}(m) & z_{33}(m) \end{vmatrix}$$

vanish. Let the s roots ($s = 6$) of the characteristic equation $\Delta = 0$ be $m_1, m_2, \cdots, m_6$. Then

$$\begin{aligned} q_1 &= C_je^{m_jt}, \\ q_2 &= C_j'e^{m_jt}, \\ q_3 &= C_j''e^{m_jt}, \end{aligned} \qquad [59]$$

[8] L. E. Dickson, *Elementary Theory of Equations*, pp. 138–149; also Vol. I, pp. 55–69.

where C_j, C'_j, C''_j are arbitrary constants in a solution of Eqs. (56) for j equal to any one of the integers from 1 to 6. Moreover,

$$\begin{aligned} q_1 &= C_1 e^{m_1 t} + C_2 e^{m_2 t} + \cdots + C_6 e^{m_6 t}, \\ q_2 &= C'_1 e^{m_1 t} + C'_2 e^{m_2 t} + \cdots + C'_6 e^{m_6 t}, \\ q_3 &= C''_1 e^{m_1 t} + C''_2 e^{m_2 t} + \cdots + C''_6 e^{m_6 t}, \end{aligned} \tag{60}$$

is a solution of Eqs. (56).

The number of arbitrary constants contained in the solution of Eqs. (56) is equal to the order of the system of the differential equations, or what is the same thing, equal to the degree of the characteristic equation $\Delta = 0$. Thus in Eqs. (60) only six of the eighteen C's are independent. It is necessary to eliminate twelve of the C's. The unprimed C's will be retained and all the primed and double-primed C's will be eliminated. Since Eqs. (59) satisfy Eqs. (56) we have, on substituting the former in the latter and dividing by $e^{m_j t}$

$$\begin{aligned} z_{11}(m_j)C_j + z_{12}(m_j)C'_j + z_{13}(m_j)C''_j &= 0, \\ z_{21}(m_j)C_j + z_{22}(m_j)C'_j + z_{23}(m_j)C''_j &= 0, \\ z_{31}(m_j)C_j + z_{32}(m_j)C'_j + z_{33}(m_j)C''_j &= 0, \end{aligned} \tag{61}$$

where $j = 1, 2, \cdots, 6$. To solve Eqs. (61) for the primed C's in terms of the unprimed C's, rewrite the equations with the unprimed C's on the right side of the equations and re-order the equations, if necessary, so that a non-vanishing determinant of order 2 appears in the upper left-hand corner of Δ, i.e.,

$$\begin{aligned} z_{12}(m_j)C'_j + z_{13}(m_j)C''_j &= -\, z_{11}(m_j)C_j, \\ z_{22}(m_j)C'_j + z_{23}(m_j)C''_j &= -\, z_{21}(m_j)C_j, \\ z_{32}(m_j)C'_j + z_{33}(m_j)C''_j &= -\, z_{31}(m_j)C_j. \end{aligned} \tag{62}$$

The first two equations of Eqs. (62) can be solved for C'_j and C''_j by Cramer's rule in terms of C_j. By a well-known algebraic theorem the values so obtained will satisfy the remaining equation. Thus

$$\begin{aligned} C'_j &= \frac{\begin{vmatrix} -z_{11}(m_j) & z_{13}(m_j) \\ -z_{21}(m_j) & z_{23}(m_j) \end{vmatrix}}{D} C_j = k'_j(m_j)C_j = (\alpha'_j + \beta'_j i)C_j \\ C''_j &= \frac{\begin{vmatrix} z_{12}(m_j) & -z_{11}(m_j) \\ z_{22}(m_j) & -z_{21}(m_j) \end{vmatrix}}{D} C_j = k''_j(m_j)C_j = (\alpha''_j + \beta''_j i)C_j \end{aligned} \tag{63}$$

where

$$D = \begin{vmatrix} z_{12}(m_j) & z_{13}(m_j) \\ z_{22}(m_j) & z_{23}(m_j) \end{vmatrix}.$$

In view of Eqs. (63), Eqs. (60) now become

$$\begin{aligned} q_1 &= C_1 e^{m_1 t} + C_2 e^{m_2 t} + \cdots + C_6 e^{m_6 t}, \\ q_2 &= k_1' C_1 e^{m_1 t} + k_2' C_2 e^{m_2 t} + \cdots + k_6' C_6 e^{m_6 t}, \\ q_3 &= k_1'' C_1 e^{m_1 t} + k_2'' C_2 e^{m_2 t} + \cdots + k_6'' C_6 e^{m_6 t}, \end{aligned} \qquad [64]$$

which contains only six arbitrary constants and is the general solution of Eqs. (56).

If all of the roots of $\Delta = 0$ are real then all the quantities in Eqs. (64) are real and the solution of the system as given by Eqs. (64) is complete.

If, on the other hand, $\Delta = 0$ has complex roots, then not all k_j' and k_j'' are real. In this case it remains to remove the apparent complex quantities from Eqs. (64). In vibration problems the roots of the characteristic equation $\Delta = 0$ are, in general, all complex. Let these roots be $m_j = -r_j \pm \omega_j i$, $(j = 1, 2, 3)$. Here, since the roots are complex the arbitrary constants in Eqs. (64) must be complex in order that q_1, q_2, q_3 be real quantities. The method of eliminating imaginary quantities from Eqs. (64) is made clear by the consideration of one pair of complex roots. Accordingly, let $m_1 = -r_1 + \omega_1 i$ and $m_2 = -r_1 - \omega_1 i$. Equations (64) then, by use of the relations

$$\begin{aligned} e^{m_1 t} &= e^{(-r_1 + \omega_1 i)t} = e^{-r_1 t}(\cos \omega_1 t + i \sin \omega_1 t), \\ e^{m_2 t} &= e^{(-r_1 - \omega_1 i)t} = e^{-r_1 t}(\cos \omega_1 t - i \sin \omega_1 t), \end{aligned}$$

become

$$\begin{aligned} q_1 &= e^{-r_1 t}[(C_1 + C_2)\cos \omega_1 t + (C_1 - C_2) i \sin \omega_1 t] + C_3 e^{m_3 t} + \cdots \\ &\quad + C_6 e^{m_6 t}, \\ q_2 &= e^{-r_1 t}[(k_1' C_1 + k_2' C_2)\cos \omega_1 t + (k_1' C_1 - k_2' C_2) i \sin \omega_1 t] \\ &\quad + C_3 k_3' e^{m_3 t} + \cdots + C_6 k_6' e^{m_6 t}, \\ q_3 &= e^{-r_1 t}[(k_1'' C_1 + k_2'' C_2)\cos \omega_1 t + (k_1'' C_1 - k_2'' C_2) i \sin \omega_1 t] \\ &\quad + C_3 k_3'' e^{m_3 t} + \cdots + C_6 k_6'' e^{m_6 t}. \end{aligned} \qquad [65]$$

If $C_1 = \dfrac{A_1 - B_1 i}{2}$, $C_2 = \dfrac{A_1 + B_1 i}{2}$ then $C_1 + C_2 = A_1$ and $(C_1 - C_2)i = B_1$, where A_1 and B_1 are real numbers. The number k_1' (Eqs. 63) is a complex number $\alpha_1' + \beta_1' i$ and it is evident from (63) that k_2' is k_1' with i replaced by $-i$. Consequently, if $k_1' = \alpha_1' + \beta_1' i$ then k_2'

$= \alpha_1' - \beta_1' i$. Substituting these values for k_1 and k_2 and the above values for C_1 and C_2 we obtain the real quantities

$$\begin{aligned} k_1'C_1 + k_2'C_2 &= \alpha_1'A_1 + \beta_1'B_1, \\ i(k_1'C_1 - k_2'C_2) &= \alpha_1'B_1 - \beta_1'A_1. \end{aligned} \tag{66}$$

If $k_1'' = \alpha_1'' + \beta_1''i$ then $k_2'' = \alpha_1'' - \beta_1''i$. In the same manner the real quantities

$$\begin{aligned} k_1''C_1 + k_2''C_2 &= \alpha_1''A_1 + \beta_1''B_1, \\ i(k_1''C_1 - k_2''C_2) &= \alpha_1''B - \beta_1''A_1. \end{aligned} \tag{67}$$

are obtained.

When the values given by Eqs. (66–67) are substituted in Eqs. (65), then q_1, q_2, q_3 are real quantities as far as the roots $-r_1 \pm \omega_1 i$ are concerned. If $r_1 = 0$, the above procedure yields the correct result, but for this a simpler procedure is given in the second illustrative example of §1·27.

To evaluate the six arbitrary constants of the solution it is necessary to know the values of q_1, q_2, q_3 and $\dot{q}_1, \dot{q}_2, \dot{q}_3$ for some value of the time.

In engineering work the frequencies of the oscillations are more often required, because of possible resonance with applied forces, than the solution of the differential equations. To obtain the frequencies of the oscillations only the solution of the characteristic equation is required since $\omega_j/2\pi$ computed from $m_j = -r_j \pm \beta_j i$, $(j = 1, 2, 3)$ gives the frequencies of the oscillations. If the characteristic equation is factorable, the roots are, of course, found by elementary methods. At all times Graeffe's method [9] yields all the roots. If there is no damping then $r_j = 0$ and the roots are pure imaginaries $\pm \omega_j i$. In this case substitute $m = \omega_j i$ in $\Delta(m) = 0$ and all the roots of the resulting equation are real. If the roots are real they can be found graphically, by guessing, or by Graeffe's or Horner's method.[10]

1·27. Illustrative Examples. Two illustrative examples are now solved; one is numerical, the other literal.

Example 1. Obtain the general solution, by the method of §1·26, of the system

$$\begin{aligned} (p^2 - 9)q_1 + (p - 1)q_2 + \quad 0\cdot q_3 &= 0, \\ (p + 3)q_1 + \quad 0\cdot q_2 + (p^2 + 16)\, q_3 &= 0, \\ 0\cdot q_1 + \quad q_2 + (p^2 + 9)\, q_3 &= 0. \end{aligned}$$

[9] J. B. Scarbough, *Numerical Mathematical Analysis*, p. 198; E. J. Berg, *Heaviside's Operational Calculus*, p. 140; also Vol. I, p. 105.

[10] L. E. Dickson, *Elementary Theory of Equations*, p. 115.

The characteristic equation is

$$\Delta = -2(p + 0.101 + 3.59i)(p + 0.101 - 3.59i)(p - 2.202)(p + 3) = 0.$$

The roots of the characteristic equations are

$$m_1 = -0.101 - 3.59i, \quad m_2 = -0.101 + 3.59i,$$
$$m_3 = -3, \qquad m_4 = 2.202.$$

The general solution is

$$q_1 = C_1 e^{m_1 t} + C_2 e^{m_2 t} + C_3 e^{m_3 t} + C_4 e^{m_4 t},$$
$$q_2 = C_1' e^{m_1 t} + C_2' e^{m_2 t} + C_3' e^{m_3 t} + C_4' e^{m_4 t},$$
$$q_3 = C_1'' e^{m_1 t} + C_2'' e^{m_2 t} + C_3'' e^{m_3 t} + C_4'' e^{m_4 t}.$$

It remains to eliminate complex quantities and the primed and double-primed constants from this solution.

Equations (62), for this example, are

$$(p - 1)C_j' + 0\cdot C_j'' = -(p - 3)(p + 3)C_j,$$
$$0\cdot C_j' + (p^2 + 16)C_j'' = -(p + 3)C_j,$$
$$C_j' + (p^2 + 9)C_j'' = 0\cdot C_j,$$

whence

$$C_j' = \frac{-\begin{vmatrix} (p-3)(p+3) & 0 \\ (p+3) & (p^2+16) \end{vmatrix}}{(p-1)(p^2+16)} C_j = -(p+3)\frac{\begin{vmatrix} (p-3) & 0 \\ 1 & 1 \end{vmatrix}}{(p-1)} C_j$$
$$= k_j' C_j = (\alpha_j' + \beta_j' i)C_j,$$

$$C_j'' = \frac{-\begin{vmatrix} (p-1) & (p-3)(p+3) \\ 0 & (p+3) \end{vmatrix}}{(p-1)(p^2+16)} C_j = -(p+3)\frac{\begin{vmatrix} 1 & (p-3) \\ 0 & 1 \end{vmatrix}}{p^2+16} C_j$$
$$= k_j'' C_j = (\alpha_j'' + \beta_j'' i)C_j.$$

In the expressions for C_j' and C_j'' let $j = 1$, i.e., $p = m_1 = -0.101 - 3.59i$. Then

$$C_1' = (-1.53 + 5.66i)C_1 = k_1' C_1 = (\alpha_1' + \beta_1' i)C_1,$$
$$C_1'' = (-0.626 + 1.295i)C_1 = k_1'' C_1 = (\alpha_1'' + \beta_1'' i)C_1.$$

Next let $j = 2$, i.e., $p = m_2 = -0.101 + 3.59i$ in C'_j and C''_j. Then

$$C'_2 = (-1.53 - 5.66i)C_2 = k'_2C_2 = (\alpha'_2 + \beta'_2 i)C_2 = (\alpha'_1 - \beta'_1 i)C_2,$$
$$C''_2 = (-0.626 - 1.295i)C_2 = k''_2C_2 = (\alpha''_2 + \beta''_2 i)C_2 = (\alpha''_1 - \beta''_1 i)C_2.$$

For $j = 3$, i.e., $p = m_3 = -3$ in C'_j and C''_j

$$C'_3 = 0\cdot C_3 = k'_3C_3 = 0, \quad C''_3 = 0\cdot C_3 = k''_3C_3 = 0.$$

For $j = 4$, i.e., $p = m_4 = 2.202$

$$C'_4 = 3.45\ C_4 = k'_4C_4, \quad C''_4 = -0.25\ C_4 = k''_4C_4.$$

Equations (66) and (67) are

$$\alpha'_1\ A_1 + \beta'_1\ B_1 = -1.53\,A_1 + 5.66\,B_1,$$
$$\alpha'_1\ B_1 - \beta'_1\ A_1 = -1.53\,B_1 - 5.66\,A_1,$$

and

$$\alpha''_1\,A_1 + \beta''_1\ B_1 = -0.626\,A_1 + 1.295\,B_1,$$
$$\alpha''_1\,B_1 - \beta''_1\ A_1 = -0.626\,B_1 - 1.295\,A_1.$$

The final substitution in Eqs. (65) gives

$$q_1 = e^{-0.101t}(A_1 \cos 3.59t + B_1 \sin 3.59t) + C_3e^{-3t} + C_4e^{2.202t},$$
$$q_2 = e^{-0.101t}[(-153A_1 + 5.66B_1) \cos 3.59t$$
$$+ (-5.66A_1 - 1.53B_1) \sin 3.59t] + 3.45C_4e^{2.202t},$$
$$q_3 = e^{-0.101t}[(-0.626A_1 + 1.295B_1) \cos 3.59t$$
$$+ (-1.295A_1 - 0.626B_1) \sin 3.59t] - 0.25C_4e^{2.202t}.$$

EXAMPLE 2. Let it be required to find the general solution of Eqs. (49) by the method of §1·26. The third and sixth equations of Eqs. (49) are independent of the remaining four and can be solved at once. The four remaining equations form two independent systems of two each, that is

$$(Mp^2 + 4k_0)X_0 - 4k_0R\eta = 0,$$
$$-4k_0RX_0 + [I_0p^2 + 4(ka^2 + k_0R^2)]\eta = 0,$$

and

$$(Mp^2 + 4k_0)Y_0 + 4k_0R\xi = 0,$$
$$4k_0RY_0 + [Ip^2 + 4(kb^2 + k_0R^2)]\xi = 0.$$

These systems are solved independently of each other. By the substitution of

$$X_0 = C_1e^{mt}, \quad \eta = C'_1e^{mt},$$

in the first system above its characteristic equation is found to be

$$\Delta = \begin{vmatrix} Mm^2 + 4k_0 & -4k_0R \\ -4k_0R & I_0m^2 + 4(ka^2 + k_0R^2) \end{vmatrix}$$
$$= I_0Mm^4 + 4(Mka^2 + Mk_0R^2 + I_0k_0)m^2 + 16k_0ka^2 = 0.$$

The four imaginary roots m_1, m_2, m_3, and m_4 of $\Delta(m) = 0$ are $\pm i\omega_1$, $\pm i\omega_2$, where

$$\omega_1 = \left[\frac{4S_1 - T_1}{2I_0M}\right]^{1/2}, \quad \omega_2 = \left[\frac{4S_1 + T_1}{2I_0M}\right]^{1/2}$$

and

$$S_1 = Mka^2 + Mk_0R^2 + I_0k_0, \quad T_1 = (16S_1^2 - 64I_0Mk_0a^2k)^{1/2}.$$

The solution of the first system is

$$X_0 = \sum_{j=1}^{j=4} C_1e^{m_jt},$$
$$\eta = \sum_{j=1}^{j=4} C_1'e^{m_jt},$$

or what is the same thing (see Eqs. 65)

$$X_0 = \sum_{j=1}^{j=2} (A_j \cos \omega_jt + B_j \sin \omega_jt),$$
$$\eta = \sum_{j=1}^{j=2} (A_j' \cos \omega_jt + B_j' \sin \omega_jt).$$

Among the eight constants of this solution only four are independent. The substitution of

$$X_0 = A_j \cos \omega_jt$$
$$\eta = A_j' \cos \omega_jt$$

in the system in question and division of the results by $\cos \omega_jt$ yield

$$(4k_0 - M\omega_j^2)A_j - 4k_0RA_j' = 0,$$
$$-4k_0RA_j + [4(ka^2 + k_0R^2) - I_0\omega_j^2]A_j' = 0.$$

Applying the theory of Eqs. (61) to the last equations we have

$$A_j' = \frac{(4k_0 - M\omega_j^2)A_j}{4k_0R} \quad (j = 1, 2).$$

In precisely the same manner the substitution of

$$X_0 = B_j \sin \omega_j t$$
$$\eta = B_j' \sin \omega_j t$$

in the same system yields eventually

$$B_j' = \frac{(4k_0 - M\omega_j^2)B_j}{4k_0 R} \quad (j = 1, 2)$$

Finally, the solution of the first system is

$$X_0 = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t + A_2 \cos \omega_2 t + B_2 \sin \omega_2 t,$$
$$\eta = \frac{(4k_0 - M\omega_1^2)}{4k_0 R}(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t) + \frac{(4k_0 - M\omega_2^2)}{4k_0 R}(A_2 \cos \omega_2 t + B_2 \sin \omega_2 t).$$

The solution of the second system is found in an identical manner and the general solution of Eqs. (49) is

$$X_0 = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t + A_2 \cos \omega_2 t + B_2 \sin \omega_2 t,$$
$$Y_0 = C_1 \cos \omega_3 t + D_1 \sin \omega_3 t + C_2 \cos \omega_4 t + D_2 \sin \omega_4 t,$$
$$Z_0 = E_1 \cos \omega_5 t + E_2 \sin \omega_5 t,$$
$$\xi = \frac{(M\omega_3^2 - 4k_0)}{4k_0 R}(C_1 \cos \omega_3 t + D_1 \sin \omega_3 t) + \frac{(M\omega_4^2 - 4k_0)}{4k_0 R}(C_2 \cos \omega_4 t + D_2 \sin \omega_4 t),$$
$$\eta = \frac{(4k_0 - M\omega_1^2)}{4k_0 R}(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t) + \frac{(4k_0 - M\omega_2^2)}{4k_0 R}(A_2 \cos \omega_2 t + B_2 \sin \omega_2 t),$$
$$\zeta = F_1 \cos \omega_6 t + F_2 \sin \omega_6 t,$$

where there are twelve arbitrary constants since the system was of order twelve and of six degrees of freedom.

EXERCISES AND PROBLEMS X

1. In the differential equations derived in Ex. 3, problem set IV, §1·10, let both the angular displacements and velocities be small. In this case the approximations $\sin \theta_1 = \theta_1$, $\cos \theta_1 = 1$, $\sin (\theta_1 - \theta_2) = \theta_1 - \theta_2$, $\dot\theta_1^2 = \dot\theta_1\dot\theta_2 = 0$, etc., can be made and the differential equations become linear with constant coefficients. Obtain the general solution of this linear system. Evaluate the arbitrary constants for the initial conditions $\theta_1(0) = \theta_0$ (small), $\theta_2(0) = \dot\theta_1(0) = \dot\theta_2(0) = 0$.

2. Obtain the general solution of the system of differential equations derived in Ex. 5, problem set IV, §1·10.

3. Obtain the general solution of the system of differential equations derived in problem 1, problem set IX, §1·24.

4. Obtain the general solution of the system of differential equations derived in problem 9, problem set IX, §1·24.

1·28. Forced Vibrations. The vibrations thus far considered are free vibrations. In contrast there exist forced vibrations, which are caused by application to the system of external forces which are functions of the time. Let the work done, in an infinitesimal displacement, by these applied forces be

$$Q_1(t)\delta q_1 + Q_2(t)\delta q_2 + \cdots + Q_n(t)\delta q_n.$$

Then Lagrange's equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r} + Q_r(t) \quad (r = 1, 2, \cdots, n) \qquad [68]$$

where V and T are given by Eqs. (54–55). In engineering work $Q_r(t)$ are developable in Fourier series

$$Q_r(t) = \sum_{s=1}^{\infty} (a_{rs} \cos s\omega_r t + b_{rs} \sin s\omega_r t).$$

1·29. Solution of Differential Equations of Forced Vibrations. The solution of Eqs. (68) consists of two parts. The first part is called the transient solution. It is obtained by solving Eqs. (68) with all $Q_r(t) = 0$. The transient solution is obtained by the method of §1·26.

It remains to obtain only the steady-state solution. First consider $Q_r(t) = 0$ for $(r = 2, 3, \cdots n)$ and $Q_1(t) = E \sin \omega t$. Write Eqs. (68)

$$\begin{aligned} z_{11}(p)q_1 + \cdots + z_{1n}(p)q_n &= \frac{E}{2i}(e^{i\omega t} - e^{-i\omega t}), \\ &\cdots\cdots\cdots\cdots \\ z_{n1}(p)q_1 + \cdots + z_{nn}(p)q_n &= 0, \end{aligned} \qquad [69]$$

where

$$z_{ij}(p) = a_{ij}\,p^2 + d_{ij}\,p + b_{ij}.$$

In solving Eqs. (69) we shall first solve

$$\begin{aligned} z_{11}(p)q_1 + \cdots + z_{1n}(p)q_n &= \frac{Ee^{i\omega t}}{2i} \\ &\cdots\cdots\cdots\cdots \\ z_{n1}(p)q_1 + \cdots + z_{nn}(p)q_n &= 0. \end{aligned} \qquad [70]$$

The substitution of $q_j = Q_j e^{i\omega t}$ $(j = 1, 2, \cdots, n)$ in Eqs. (69) yields

$$\begin{aligned} \lambda_{11}Q_1 + \cdots + \lambda_{1n}Q_n &= \frac{E}{2i} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_{n1}Q_1 + \cdots + \lambda_{nn}Q_n &= 0, \end{aligned} \tag{71}$$

where

$$\lambda_{ij} = -a_{ij}\omega^2 + id_{ij}\omega + b_{ij}.$$

The solution, by determinants, of Eqs. (71) for any Q (say Q_k) is

$$Q_k = \frac{E\,A_{1k}(i\omega)}{2i\Delta(i\omega)}, \tag{72}$$

where $\Delta(i\omega)$ is the determinant of the coefficients of Q_j in the system (71) and $A_{1k}(i\omega)$ is the cofactor of λ_{1k} in $\Delta(i\omega)$. If $\Delta(i\omega)/A_{1k}(i\omega)$ is denoted by $Z_{1k}(i\omega)$ then

$$q_k = \frac{E\,e^{i\omega t}}{2i\,Z_{1k}(i\omega)}. \tag{73}$$

To obtain the solution of Eqs. (70) with $\dfrac{E\,e^{i\omega t}}{2i}$ replaced by $+\dfrac{E\,e^{-i\omega t}}{2i}$ it is necessary only to replace i by $-i$ in Eq. (73). Then the steady-state solution of Eqs. (69) is the sum of (73) and (73) with i replaced by $-i$, or

$$q_k = \frac{E}{2i}\left[\frac{e^{i\omega t}}{Z_{1k}(i\omega)} - \frac{e^{-i\omega t}}{Z_{1k}(-i\omega)}\right]. \tag{74}$$

Since $\Delta(i\omega)$ and $A_{1k}(i\omega)$ are both polynomials in $i\omega$, both are complex numbers. Hence $Z_{1k}(i\omega)$ is a complex number (say $a + bi$). The complex number $a + bi$ can be written $re^{i\varphi}$ where r is its modulus and φ its argument. But since $Z_{1k}(-i\omega)$ is obtained from $Z_{1k}(i\omega)$ by replacing i by $-i$, it follows that $Z_{1k}(-i\omega)$ and $Z_{1k}(i\omega)$ are conjugate complex numbers. Thus if

$$\begin{aligned} Z_{1k}(i\omega) &= re^{i\varphi_k}, \\ Z_{1k}(-i\omega) &= re^{-i\varphi_k}. \end{aligned}$$

When these values for $Z_{1k}(i\omega)$ and $Z_{1k}(-i\omega)$ are substituted in Eqs. (74)

$$q_k = \frac{E}{2ri}\left[e^{i(\omega t-\varphi_k)} - e^{-i(\omega t-\varphi_k)}\right] = \frac{E\sin(\omega t - \varphi_k)}{|Z_{1k}(i\omega)|}. \tag{75}$$

EXAMPLE. Obtain the complete solution of the system of differential equations

$$(p^2 - 9)q_1 + (p - 1)q_2 + 0 \cdot q_3 = 3 \sin 5t,$$
$$(p + 3)q_1 + 0 \cdot q_2 + (p^2 + 16)q_3 = 0,$$
$$0 \cdot q_1 + q_2 + (p^2 + 9)q_3 = 0.$$

The complementary function or transient solution is given in §1·27. It remains to find the particular integral or steady-state solution. From Eq. 75

$$q_1 = \frac{3 \sin (5t - \varphi_1)}{|Z_{11}(5i)|}, \quad q_2 = \frac{3 \sin (5t - \varphi_2)}{|Z_{12}(5i)|}, \quad q_3 = \frac{3 \sin (5t - \varphi_3)}{|Z_{13}(5i)|},$$

where

$$\Delta(5i) = \begin{vmatrix} (5i)^2 - 9 & 5i - 1 & 0 \\ (5i) + 3 & 0 & (5i)^2 + 16 \\ 0 & 1 & (5i)^2 + 9 \end{vmatrix} = -754 + 160i,$$

$$A_{11}(5i) = 9, \quad A_{12}(5i) = 16(3 + 5i), \quad A_{13} = (5i + 3),$$

$$Z_{11}(5i) = -83.8 + 17.8i, \quad Z_{12}(5i) = -2.69 + 7.81i,$$

$$Z_{13}(5i) = -43 + 125i,$$

$$\varphi_1 = \tan^{-1} \frac{17.8}{-83.8} = 168°, \quad \varphi_2 = \tan^{-1} \frac{7.8}{-2.69} = 108°,$$

$$\varphi_3 = \tan^{-1} \frac{125}{-43} = 108°.$$

Finally,

$$q_1 = \frac{3}{85.7} \sin (5t - 168°), \quad q_2 = \frac{3}{8.26} \sin (5t - 109°),$$

$$q_3 = \frac{3}{132} \sin (5t - 109°).$$

Let the above values of q_1, q_2, q_3 be denoted by q_{1s}, q_{2s}, q_{3s} and those of §1·26 be denoted by q_{1t}, q_{2t}, q_{3t}. Then the complete solution of the illustrative example is

$$q_1 = q_{1t} + q_{1s}, \quad q_1 = q_{2t} + q_{2s}, \quad q_3 = q_{3t} + q_{3s}.$$

1·30. More General $Q_s(t)$ and Resonance. If $Q_s(t) = E \sin s\omega t$ and all other Q's are zero then the steady-state solution of §1·29 is given by replacing ω throughout by $s\omega$. If $Q_s(t) = E \cos \omega t$, then the solution is given by Eqs. (75) but with $\sin (\omega t - \varphi_k)$ replaced by $\cos (\omega t - \varphi_k)$. If $Q_r(t)$ is a Fourier series the steady-state solution is the

sum of the separate solutions obtained by employing sequentially the terms of the Fourier series.

Suppose next that no $Q_r(t)$ is zero. The procedure is as follows: First solve Eqs. (68) under the restriction that $Q_r(t) = 0$, $(r = 2, 3, \cdots, n)$ and $Q_1(t) \neq 0$. Next let $Q_r(t) = 0$, $(r = 1, 3, \cdots, n)$ and $Q_2(t) \neq 0$. Carry on this process, finally solving Eqs. (68), for all $Q_r(t) = 0$ except $Q_n(t)$. The n values obtained for q_k are then added giving the complete steady-state solution for q_k.

If the number of dependent variables is large it is more convenient to abandon the classical method of solution of §§1·26–1·29 and to resort to operational methods.[11]

If, in computing the steady-state solution, $\Delta(i\omega) = 0$ then **resonance** is said to exist between the applied force or voltage $E \sin \omega t$ and the system on which the force or voltage acts. In this case Eq. (75) does not give the steady-state solution. In fact the resonance solution will contain t at least linearly.

1·31. Normal Coordinates. The potential and kinetic energies of a vibrational system are both definite quadratic forms in $q_1, q_2, \cdots, q_n$ and $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ respectively. By a well-known algebraic theorem [12] there exists a real linear transformation of coordinates and velocities $q_1, q_2, \cdots, q_n$ and $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$ which changes Eqs. (54–55) to the forms

$$V = \tfrac{1}{2}(\mu_1\xi_1^2 + \mu_2\xi_2^2 + \cdots + \mu_n\xi_n^2), \qquad [76]$$

$$T = \tfrac{1}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2 + \cdots + \dot{\xi}_n^2), \qquad [77]$$

where $\mu_1, \mu_2, \cdots, \mu_n$ are real constants. The coordinates $\xi_1, \xi_2, \cdots, \xi_n$ are called **normal** or **principal coordinates** of the vibrating system. In these new coordinates Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_k}\right) = -\frac{\partial V}{\partial \xi_k} \quad (k = 1, 2, \cdots, n) \qquad [78]$$

or

$$\ddot{\xi} + \mu_k\xi = 0 \quad (k = 1, 2, \cdots, n). \qquad [79]$$

The solutions of the n independent differential equations are

$$\xi_k = A_k \sin \sqrt{\mu_k}(t - \alpha_k) \quad (k = 1, 2, \cdots, n). \qquad [80]$$

The solutions (80) are simple. However, the linear transformation reducing V and T to the forms (76) and (77) is tedious and involves a knowledge of the roots of the characteristic equation. The natural frequencies of the vibrations are the same as already obtained in §1·26.

[11] Vol. I, Chap. IV.

[12] L. E. Dickson, *Modern Algebraic Theories*, p. 74; E. T. Whittaker, *Analytical Dynamics*, p. 181.

EXERCISES XI

1. Obtain the steady-state solution of the illustrative example of §1·29 with $3 \sin 5t$ replaced by $5 \sin 3t$.
2. Obtain the complete solution of the system

$$(p^2 + 4)q_1 + \sqrt{13}q_2 = 5 \sin 2t,$$
$$-\sqrt{13}q_1 + (p + 1)q_2 = 0.$$

3. Obtain the complete solution of Ex. 1, set VI, Chap. I.
4. Obtain the complete solution of Ex. 3, set VI, Chap. I.
5. Write the differential equations of motion and obtain the complete solution of problem 5, set IX, Chap. I.
6. Obtain the complete solution of problem 6, set IX, Chap. I.

1·32. Electric Locomotive Oscillations. As a general example illustrating both the dynamical principles thus far developed and the method of engineering analysis described in the introduction of this textbook, the motions of an electric locomotive are analyzed.[13]

(*a*) *Factual information.* Experience classifies the five oscillatory motions of an electric locomotive as pitch, roll, plunge, nose, and rear-end lash. The last two are especially important because their pronounced existence in a locomotive signifies a tendency to derail. Considered superficially, characteristic oscillations of an electric locomotive would seem to be very similar to those of an ordinary vehicle such as an automobile, but experimental data and observation indicate the existence of dangerous nose and rear-end lash which are not oscillations common to an automobile. If the tendency to nose exists in an electric locomotive and if the locomotive noses for a given speed V_0 then it will nose more violently for *all* speeds greater than V_0. Consequently, nosing is not a resonance phenomenon and cannot be avoided by running at a slightly different speed. It might be supposed that nosing is due to the coning of the wheels or to the staggering of the rails or to a combination of these two possible causes. However, such causes would produce resonance frequencies for definite discrete values of V instead of instability for all values of V exceeding V_0. Rails on European railroads are not staggered and yet locomotive nosing persists. The tendency to nose and the violence of the oscillation increase with the weight and power of the locomotive. Nosing usually starts as a roll induced by the locomotive rounding a curve onto straight track, but unlike the oscillations of roll, pitch, and plunge, once it is set up it is not damped until the speed of the locomotive is reduced. The pulling of a train has only a second order effect on the nosing of a locomotive.

[13] From unpublished work of B. S. Cain and E. G. Keller.

This dangerous oscillation of a locomotive occurs most frequently on straight track at high constant speed. When rounding a curve the flanges of the wheels remain in contact with the outside rail and nosing is not pronounced.

(*b*) *Theory of performance.* The postulated theory of performance is that the energy of nosing oscillation is transferred from the motors of the locomotive to the mass of the locomotive through the creepage action of the driving wheels.

(*c*) *Assumptions.* It is assumed that (1) impacts can be replaced by continuous forces acting through finite intervals of time; (2) the driving wheels roll and creep, but do not slide; (3) the creepage forces are functions of the velocities and displacements.

(*d*) *Choice of principles.* The derivation of the differential equations is based on Lagrange's equations of dynamics.

(*e*) *Derivation of the equations of motion.* Although the method can be extended to locomotives of any type, we shall for simplicity set up the differential equations of motion for locomotives of type 2—C—2. (Two-axle guiding truck — three driving axles — two-axle rear truck.) The three groups of forces acting on the spring-borne mass of a locomotive are (1) spring, (2) creepage, (3) flange, and (4) damping forces.

(1) *Spring forces.* The spring arrangement of the 2—C—2 type is the same as that of the 2—B—2 locomotive described in §1·21 and its potential energy is given by the last equation of the same article.

(2) *Creepage forces.* The action of a locomotive driving wheel, because of the creep of metal at the region of contact of wheel and rail, is not one of simple rolling. Instead, forces exist at the treads of the two wheels of a driving axle, which, if referred to the center of the driving axle, constitute a torque about a line through the center of the axle and perpendicular to the plane of the track and lateral and longitudinal forces acting at the same point. A creepage force F is defined by the equation

$$F = -fd, \tag{81}$$

where

$$d = \frac{\text{displacement} - \text{rolling displacement}}{\text{rolling displacement}} \tag{82}$$

and f is the coefficient of creepage which is calculated by the formula

$$f = A(rW)^{1/2}.$$

In this formula r is the radius of a driving wheel in millimeters, W is the weight expressed in kilograms borne by one wheel, and A is an empirical constant equal to 800.

Let the following symbols have the significance indicated:

$2b$ = track gage,
$2\delta_1$ = lateral play between flanges of the driving wheels and rails,
$2\delta_2$ = lateral play between flanges of guiding trucks and the rails,
r = radius of driving wheel,
λ = tangent of the angle of coning of tire,
θ = angle through which a wheel has turned in a rolling displacement,
φ = angle the driving axles make at any time with the horizontal perpendicular to the track or the angle the frame makes with the center line of the track (Fig. 1·34),
(x, y) = coordinates of center of driving axle (Fig. 1·34).

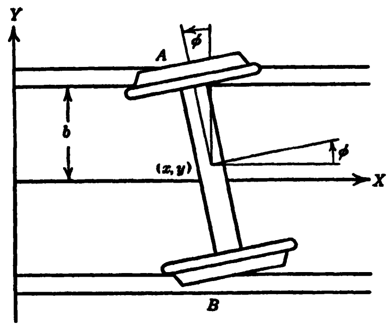

FIG. 1·34. Creepage Displacements for Driving Wheels of Electric Locomotive.

The meaning of h_1, h_2, b_1, b_2, and $2c$ is given in §1·21.

To obtain the force F it is necessary only to compute d by the substitution of the various displacements in Eq. (82). Let fixed axes be taken as indicated in Fig. 1·34. Let A and B denote the points of contact of the driving wheels with the rails. The coordinates of A and B are

$$(A)\quad (x - b\varphi, y + b),$$
$$(B)\quad (x + b\varphi, y - b).$$

The rolling displacements, to the accuracy required, are

$$(A)\quad [(r + y)d\theta, r\varphi\, d\theta],$$
$$(B)\quad [(r - y)d\theta, r\varphi\, d\theta].$$

The components of creepage at A and B are

$$(A)\left[\frac{dx}{rd\theta} - 1 - \left(\frac{b}{r}\frac{d\varphi}{d\theta} + \lambda\frac{y}{r}\right), \frac{dy}{rd\theta} - \varphi\right],$$
$$(B)\left[\frac{dx}{rd\theta} - 1 + \left(\frac{b}{r}\frac{d\varphi}{d\theta} + \lambda\frac{y}{r}\right), \frac{dy}{rd\theta} - \varphi\right].$$

The component forces are

$$(A)\quad -f\left[\frac{dx}{rd\theta} - 1 - \left(\frac{b}{r}\frac{d\varphi}{d\theta} + \lambda\frac{y}{r}\right)\right], -f\left[\frac{dy}{rd\theta} - \varphi\right],$$
$$(B)\quad -f\left[\frac{dx}{rd\theta} - 1 + \left(\frac{b}{r}\frac{d\varphi}{d\theta} + \lambda\frac{y}{r}\right)\right], -f\left[\frac{dy}{rd\theta} - \varphi\right].$$

The component forces acting at A and B are equivalent to the torque G_1 and forces X_1 and Y_1 acting at the center of the driving axle

$$G_1 = -2f\left(\frac{b^2}{r}\frac{d\varphi}{d\theta} + \frac{\lambda b}{r}\right)y,$$

$$X_1 = -2f\left(\frac{dx}{rd\theta} - 1\right),$$

$$Y_1 = -2f\left(\frac{dy}{rd\theta} - \varphi\right).$$

If V is the constant forward velocity of progression then $V\,dt \doteq rd\theta$ since f is extremely large in comparison with X_1. The last equations, in view of this approximation, are

$$\begin{aligned} G_1 &= -2f\left(\frac{b^2\dot{\varphi}}{V} + \frac{\lambda b}{r}\right)y, \\ X_1 &= \text{a constant}, \\ Y_2 &= -2f\left(\frac{\dot{y}}{V} - \varphi\right). \end{aligned} \qquad [83]$$

The second of Eqs. (83) implies constant forward velocity which is the only case of interest. All driving axles are attached rigidly to the frame of the locomotive with the exception that vertical motion of the frame with respect to the axles is possible. Equations (83) are to be summed over all driving axles.

(3) *Flange forces.* The flange forces F_1, F_2, f_1, and f_2, which act at the points N_1, N_2, N_3, and N_4 shown in Fig. 1·35, are non-linear functions of the displacements of the points of application. To the accuracy required

$$\begin{aligned} F_1 &= H_1\left[\frac{y_1}{\delta_1}\right]^3 + I_1\left[\frac{y_1}{\delta_1}\right]^5 + J_1\left[\frac{y_1}{\delta_1}\right]^7 + \cdots \\ F_2 &= H_1\left[\frac{y_2}{\delta_1}\right]^3 + I_1\left[\frac{y_2}{\delta_1}\right]^5 + J_1\left[\frac{y_2}{\delta_1}\right]^7 + \cdots \\ f_1 &= h_1\left[\frac{y_3}{\delta_2}\right]^3 + i_1\left[\frac{y_3}{\delta_2}\right]^5 + j_1\left[\frac{y_3}{\delta_2}\right]^7 + \cdots \\ f_2 &= h_1\left[\frac{y_4}{\delta_2}\right]^3 + i_1\left[\frac{y_4}{\delta_2}\right]^5 + j_1\left[\frac{y_4}{\delta_2}\right]^7 + \cdots \end{aligned} \qquad [84]$$

where y_1, y_2, y_3, and y_4 are displacements of the center points of the driving axles and guiding trucks from a vertical plane passing through the center line of the track. The flange forces on the middle driver can be neglected. The constants H_1, I_1, J_1, h_1, i_1, j_1 are determined from force curves.

Inspection of Figs. 1·35–1·36 and use of Eqs. (83–84) yield the following table of creepage and flange forces. The locomotive is supposed displaced in the direction of positive φ and the Roman numerals refer to the parts, shown in Fig. 1·36, of the locomotive on which the forces and torques act.

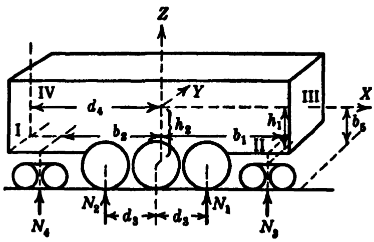

FIG. 1·35. Dimensions for Electric Locomotive of Type 2–C±2.

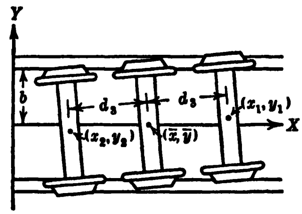

FIG. 1·36. Driving Truck for Electric Locomotive of Type 2–C±2.

	I	II	III	IV
X:	0	0	0	0
Y:	$-F_2 - f_2 - 2f\left(\frac{\dot{y}_2}{V} - \varphi\right)$ $- 2f\left(\frac{\dot{\bar{y}}}{V} - \varphi\right)$	0	$-F_1 - f_1 - 2f\left(\frac{\dot{y}_1}{V} - \varphi\right)$	0
Z:	b 0	0	0	0
Ξ:	$b_5\Big[-F_2 - f_2 - 2f\left(\frac{\dot{y}_2}{V} - \varphi\right)$ $- 2f\left(\frac{\dot{\bar{y}}}{V} - \varphi\right)\Big]$	0	$b_5\left[-F_1 - f_1 - 2f\left(\frac{\dot{y}_1}{V} - \varphi\right)\right]$	0
H:	0	0	0	0
Z:	$d_3F_2 + d_4f_2 + 2fd\left(\frac{\dot{y}_2}{V} - \varphi\right)$ $- 2f\left(\frac{b^2}{V}\dot{\varphi} + \frac{\lambda b}{r}y_2\right)$ $- 2f\left(\frac{b^2}{V}\dot{\varphi} + \frac{\lambda b}{r}\bar{y}\right)$	0 0 0	$- d_3F_1 - d_4f_1 - 2df\left(\frac{\dot{y}_1}{V} - \varphi\right)$ $- 2f\left(b^2\dot{\varphi} + \frac{\lambda}{r}by_1\right)$ $+ F_1(y_1)$	0 0 0

Let the origin of coordinates XYZ be a point in the vertical plane passing through the center line of the track. This point is at the height of the center of gravity and has the same forward velocity as the locomotive. When the locomotive is in equilibrium position the center of gravity coincides with the origin and x_0, y_0, z_0, ξ, η, and ζ all vanish. It should be noted that, because of the constraints of the journals, ζ of Fig. 1·27 is identical to φ of Fig. 1·34. Moreover, x_0, y_0, z_0 of § 1·32 are identical to X_0, Y_0, Z_0 of § 1·21.

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r} + Q_r^{(0)} \quad (r = 1, 2, \cdots, 6),$$

where

$$q_1 = x_0, \quad q_2 = y_0, \quad q_3 = z_0, \quad q_4 = \xi, \quad q_5 = \eta, \quad q_6 = \zeta,$$

and $Q_r^{(0)}$ are the forces given in the preceding table, and T and V are given in §§1·20–1·21. The complete differential equations of the problem are

$$\begin{aligned}
M\ddot{x}_0 &= 0,\\
M\ddot{y}_0 &= -F_2 - f_2 - 2f\left(\frac{\dot{y}_2}{V} - \varphi\right) - 2f\left(\frac{\dot{\bar{y}}}{V} - \varphi\right)\\
&\qquad\qquad -F_1 - f_1 - 2f\left(\frac{\dot{y}_1}{V} - \varphi\right).\\
M\ddot{z}_0 &+ \lambda_1(z_0 - b_1\eta) + \lambda_2(z_0 + c\xi + b_2\eta)\\
&\qquad\qquad + \lambda_2(z_0 - c\xi + b_2\eta) + k_1^2\dot{z}_0 = 0,\\
A\ddot{\xi} &+ \lambda_2 c(z_0 + c\xi + b_2\eta) - \lambda_2 c(z_0 - c\xi + b_2\eta) + k_2^2\xi =\\
&\quad -b_5(F_1 + F_2 + f_1 + f_2) - 2b_5\frac{f}{V}(\dot{y}_1 + \dot{\bar{y}} + \dot{y}_2) + 6b_5 f\varphi,\\
B\ddot{\eta} &- \lambda_1 b_1(z_0 - b_1\eta) + \lambda_2 b_2(z_0 + c\xi + b_2\eta)\\
&\qquad\qquad + \lambda_2 b_2(z_0 - c\xi + b_2\eta) + k_3^2\dot{\eta} = 0,\\
C\ddot{\zeta} &= -d_3(F_1 - F_2) - d_4(f_1 - f_2) - \frac{2fd_3}{V}(\dot{y}_1 - \dot{y}_2)\\
&\qquad\qquad - \frac{6fb^2}{V}\dot{\zeta} - \frac{2f\lambda b}{r}(\bar{y} + y_1 + y_2) + F_1(\bar{y}_1).
\end{aligned} \qquad [85]$$

The number of dependent variables in the differential equations is 11, but y_1, y_2, y_3, y_4, and $\bar{y}$ are expressible in terms of y_0, ξ, and ζ by means of the relations

$$y_1 = \mathbf{j}\cdot\mathbf{s}_0 + \mathbf{j}\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \xi & \eta & \zeta \\ d_3 & 0 & -b_5 \end{vmatrix} = y_0 + b_5\xi + d_3\zeta,$$

$$y_2 = y_0 + b_5\xi - d_3\zeta, \quad y_3 = y_0 + h_1\xi + d_4\zeta,$$

$$y_4 = y_0 + h_2\xi - d_4\zeta, \quad \bar{y} = y_0 + b_5\xi.$$

Thus the number of dependent variables of Eqs. (85) is reduced to six which is the number of differential equations of the system. The points of application of F_1, F_2, f_1, and f_2 are taken with sufficient accuracy to be points in the plane of the track and directly beneath either the midpoints of the driving axles or the center points of the guiding trucks.

(4) *Damping Forces.* In an electric locomotive there are two kinds of mechanical damping forces, structural and creepage. The latter are functions of the speed; the former are not. Motion is stable or unstable according as the total damping is positive or negative.

(*f*) *Solution of the system of differential equations.* Equations of the form of Eqs. (85) are solvable by the methods of Chap. III and in particular by Cotton's method indicated in Ref. 11 of Chap. III. The only purpose of the solution of the differential equations is a check on the theory of performance because a useful and simpler criterion of the stability of the locomotive is obtainable by very little labor.

The differential equations (85) are non-linear equations, the non-linearity being introduced by the flange forces. Derailment of the locomotive is, of course, prevented only by the flanges. Yet the motion defined by the linear terms of Eqs. (85) may be either stable or unstable. If any roots of the characteristic equation of Eqs. (85) with $F_1 = F_2 = f_1 = f_2 = 0$ possess positive real parts the motion is unstable and the locomotive is said to be unstable. The roots of the characteristic equation are a function of V. The problem is thus reduced to so specifying the constants of the locomotive (particularly spring constants) that the roots in question do not possess positive real parts except for excessively large values of V.

It is unnecessary to solve the characteristic equation since there exists a criterion by which it is possible to determine the number of roots of a characteristic equation which have positive real parts without obtaining these roots.[14]

[14] E. J. Routh, *Advanced Rigid Dynamics*, p. 170; or Vol. I, p. 129.

It is beyond the present purpose to solve Eqs. (85). The calculation of the characteristic equation for the motion described by the linear terms is left as Ex. 1.

(*g*) *Experimental checks.* The periods of oscillations calculated from the solutions of Eqs. (85) were approximately checked experimentally by test runs on the Erie test tracks of the General Electric Company. Confidence was gained in the theory of performance which was postulated.

EXERCISES AND PROBLEMS XII

1. Compute the characteristic equation for Eqs. (85) with $F_1 = F_2 = f_1 = f_2 = 0$.
2. Is the variation in the height of the center of gravity during motion taken into account in computing the potential energy of the locomotive in §1·21?
3. Derive the differential equations of motion for the more complicated locomotive of Fig. 1·33.
4. Develop another mathematical theory of locomotive oscillations which takes into account impacts between wheel flanges and rails. (Consult Ref. 6 at end of chapter for Lagrange's equations and impulsive motion).

(5)

Lagrange's Equations and Holonomic Systems

The dynamical systems analyzed thus far possessed precisely the same number of degrees of freedom as there were dependent variables in Lagrange's equations. That is, the system possessed n degrees of freedom. In a more general situation m relations exist between $q_1, q_2, \cdots, q_n$ in addition to the differential equations of Lagrange. These relations are expressed by Eqs. (28). If Eqs. (28) are integrable then the dynamical system is said to be **holonomic,** if not, it is said to be **non-holonomic.**

1·33. Modification of Lagrange's Equations for Holonomic Systems. Let the m constraints be expressed by the equations

$$C_{k1}\,\delta q_1 + C_{k2}\,\delta q_2 + \cdots + C_{kn}\,\delta q_n = 0 \quad (k = 1, 2, \cdots, m) \qquad [86]$$

where the C's are functions of $q_1, q_2, \cdots, q_n$. In this section (86) are integrable. Thus the dynamical system possesses exactly n-m degrees of freedom. From Eqs. (30) we have

$$\sum_{r=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} - Q_r\right]\delta q_r = 0. \qquad [87]$$

Multiplying the first, second, etc., of (86) respectively by the undeter-

mined multipliers $\lambda_1, \lambda_2, \cdots, \lambda_m$ and adding the results to (87) we have

$$\sum_{r=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} - Q_r + \sum_{i=1}^{m} \lambda_i C_{ir}\right] \delta q_r = 0. \qquad [88]$$

The m Eqs. (86) contain the n unknowns $\delta q_1, \delta q_2, \cdots, \delta q_n$. From the theory [15] of such equations the values of n-m unknowns (say $\delta q_{m+1}, \cdots, \delta q_n$) can be assigned arbitrary values and the equations then solved for $\delta q_1, \cdots, \delta q_m$. Next let the m undetermined multipliers $\lambda_1, \cdots, \lambda_m$ be chosen so that the m equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} - Q_r + \lambda_1 C_{1r} + \cdots + \lambda_m C_{mr} = 0$$

$$(r = 1, 2, \cdots, m) \qquad [89]$$

are satisfied. Then Eq. (88) reduces to

$$\sum_{r=m+1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} - Q_r + \lambda_1 C_{1r} + \cdots + \lambda_m C_{mr}\right] \delta q_r = 0. \qquad [90]$$

If $\delta q_{m+1} =$ constant $\neq 0$ and $\delta q_{m+2} = \delta q_{m+3} = \cdots = \delta q_n = 0$ then (90) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_{m+1}}\right) - \frac{\partial T}{\partial q_{m+1}} - Q_{m+1} + \lambda_1 C_{1\,m+1} + \cdots + \lambda_m C_{m\,m+1} = 0. \qquad [91]$$

If $\delta q_{m+2} =$ constant $\neq 0$ and $\delta q_{m+1} = \delta q_{m+3} = \cdots = \delta q_n = 0$ then (91) is obtained with $m + 1$ replaced by $m + 2$. Continuing this process n-m equations similar to (91) are obtained. These n-m equations, along with (89), form the system of n equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} - Q_r + \lambda_1 C_{1r} + \cdots + \lambda_m C_{mr} = 0$$

$$(r = 1, 2, \cdots, n.) \qquad [92]$$

When the n-m multipliers $\lambda_1, \lambda_2, \cdots, \lambda_m$ have been eliminated from (92) n-m equations in $q_1, q_2, \cdots, q_r$ remain. These equations along with the m Eqs. (86) furnish n equations for the determination of $q_1, q_2, \cdots, q_n$.

Example 1. A homogeneous and perfectly rough sphere of mass m and radius r rolls on a fixed sphere of radius R. The only external force is gravity. Obtain the differential equation of motion.

[15] Vol. I, p. 64.

Let the coordinates and dimensions be represented in Fig. 1·37. Evidently,

$$T = \frac{m}{2}\left[(r + R)^2\dot{q}_2^2 + \tfrac{2}{5}r^2\dot{q}_1^2\right],$$

$$V = mg(r + R)\cos q_2.$$

Since the contact is rough, δq_1 and δq_2 are not independent. To obtain the relation between δq_1 and δq_2 it is necessary only to note that, at the point of contact of the two spheres,

$$r\dot{q}_1 = R\dot{q}_2$$

from which, by integration

$$rq_1 = Rq_2.$$

From the last equation, by taking variations, the equation corresponding to Eqs. (86) is

$$r\,\delta q_1 = R\,\delta q_2,$$

where $C_{11} = r$ and $C_{12} = -R$. The equations corresponding to Eqs. (89) and (91) are respectively

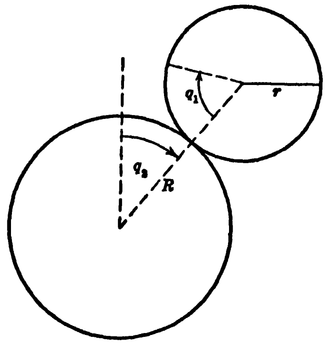

FIG. 1·37

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} + \lambda_1 r = 0,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} - \lambda_1 R = 0.$$

These equations correspond to Eqs. (92).

Eliminating λ_1 between the last two equations and substituting the values of T and V we obtain

$$(2/5)rR\ddot{q}_1 + (r + R)^2\ddot{q}_2 - (r + R)g\sin q_2 = 0.$$

After q_1 has been eliminated, by means of the relation $rq_1 = Rq_2$, the final equation is

$$[(2/5)R^2 + (r + R)^2]\ddot{q}_2 - (r + R)g\sin q_2 = 0.$$

EXERCISES AND PROBLEMS XIII

1. A hemisphere rocks on a rough plane. Obtain the differential equation of motion using the coordinates θ and x_0 shown in Fig. 1·38*a*.

2. The flywheel, rods, and horizontal piston represented in Fig. 1·38*b* assume an equilibrium position when there is no steam in the cylinder. Taking q_1 and q_2 as

generalized coordinates, obtain the differential equation of motion of the system when displaced from equilibrium position. (NOTE: δq_1 and δq_2 are not independent and the problem has one equation of constraint.) Show that if the engine is statically balanced it is not dynamically balanced.

3. Obtain Lagrange's equations of motion for the governor represented in Fig. 1·38*c*. Employ as coordinates the angles θ and φ shown. (NOTE: the problem involves no constraints.) *Hint:*

$$T = C\dot{\theta}^2/2 + I\dot{\varphi}^2/2,$$

where C and I are functions of θ and I includes the moment of inertia of both the engine and the machinery driven. Denote the potential energy of the governor by

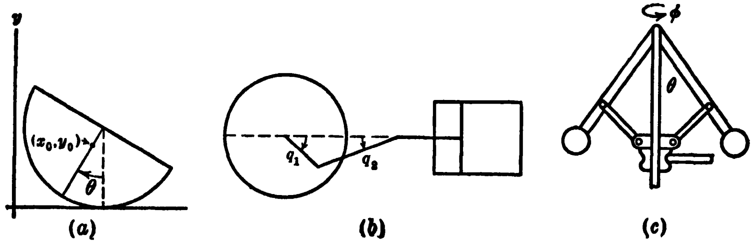

FIG. 1·38

V and let Φ be the generalized force representing the excess driving torque over resistance.

(6)

Non-holonomic Systems

The dynamical systems of this section differ from those of Sec. 5 only in the nature of the constraints. The m equations of constraint

$$C_{k1}\,\delta q_1 + C_{k2}\,\delta q_2 + \cdots + C_{kn}\,\delta q_n + T_k\,dt = 0 \qquad (k = 1, 2, \cdots, m) \quad [93]$$

are *non*-integrable and thus the system considered retains n degrees of freedom corresponding to the n-coordinates $q_1, q_2, \cdots, q_n$. Non-holonomic systems can be regarded as holonomic systems by taking into consideration certain reactions of the constraints.

1·34. Reduction to Holonomic Form. To the generalized forces $Q_1, Q_2, \cdots, Q_n$ of Eqs. (31) let there be added n additional generalized forces $Q_1', Q_2', \cdots, Q_n'$. The latter are forces exerted by the constraints which compel the system to fulfil the kinematical conditions of the dynamical system. The constraints may now be considered removed

and replaced by the forces $Q'_1, Q'_2, \cdots, Q'_n$. Consequently, the system is now holonomic and the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r + Q'_r \quad (r = 1, 2, \cdots, n). \qquad [94]$$

It remains to describe the generalized forces Q'_r. The equation

$$Q'_1\,\delta q_1 + Q'_2\,\delta q_2 + \cdots + Q'_n\,\delta q_n = 0 \qquad [95]$$

and the m equations

$$C_{k1}\,\delta q_1 + C_{k2}\,\delta q_2 + \cdots + C_{kn}\,\delta q_n = 0 \quad (k = 1, 2, \cdots, m) \qquad [96]$$

state that the work done (left member of Eq. 95) by the additional forces of constraint in displacements permissible by the constraints (Eq. 95) is zero. Multiplying the first, second, etc., of Eqs. (96) respectively by the undetermined multipliers $\lambda_1, \lambda_2, \cdots, \lambda_m$ and adding the results to (87) where Q_r has been replaced by $Q_r + Q'_r$ we obtain

$$\sum_{r=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} - Q_r - Q'_r + \sum_{i=1}^{m}\lambda_i C_{ir}\right]\delta q_r = 0. \qquad [97]$$

By means of (95) Eq. (97) reduces to (88). The reasoning from (88) to (92) of Sec. 5 is repeated.

When the m multipliers $\lambda_1, \lambda_2, \cdots, \lambda_m$ have been eliminated from Eq. (92) then n-m equations in $q_1, q_2, \cdots, q_n$ remain. These equations along with the m equations

$$C_{k1}\dot{q}_1 + C_{k2}\dot{q}_2 + \cdots + C_{km}\dot{q}_m + T_k = 0,$$

furnish n equations for the determination of $q_1, q_2, \cdots, q_n$.

(7)

Energy Method and Rayleigh's Principle

In Sec. 4, §1·26 and §1·31, two methods are given for obtaining the natural periods of vibration of an elastic system with a finite number of degrees of freedom. The labor involved by either method is considerable; in the first it is necessary to solve the characteristic equation $\Delta = 0$; in the second the successive transformations introducing normal coordinates are required. Rayleigh's principle is frequently not only more easily applied, but it is also applicable to continuous systems with infinitely many degrees of freedom.

1·35. General and Normal Modes of Vibration. The simultaneous Eqs. (80), §1·31, describe in normal coordinates the most general vibration of an elastic system possessing n degrees of freedom. Seldom are

the most general vibrations of interest. Instead there exist natural or normal modes of vibration characterized by the fact that the motion of each particle is simply periodic and given by one of Eqs. (80); the other A_k being zero. There are thus, in general, n distinct normal modes. The mode of lowest (smallest) frequency is called the **fundamental mode.** The frequencies of the normal modes are called **natural frequencies.** The smallest of these is the **fundamental frequency.** A frequency when multiplied by 2π is called a **pulsatance.** By the introduction of frictionless constraints (consider one side of the motor analyzed in §1·20 to be constrained by a hinge) each particle of an elastic system can be compelled to vibrate with frequency $\omega/2\pi$ or pulsatance ω according to the equation $q_i = B_i \sin \omega t$, where ω is not necessarily a natural pulsatance of the system.

1·36. Energy Method for Systems with a Finite Number of Degrees of Freedom. This method gives the n natural frequencies. Let the holonomic conservative elastic system be specified by the coordinates $q_1, q_2, \cdots, q_n$ and the potential and kinetic energies be given by Eqs. (54–55) respectively. Let the system describe, by introduction of frictionless constraints, simply periodic motion according to the equations

$$q_i = x_i \cos \omega t \quad (i = 1, 2, \cdots, n) \qquad [98]$$

where ω is, in general, not a natural pulsatance. If Eqs. (98) are substituted in Eqs. (54–55) then

$$V = \tfrac{1}{2}(b_{11}x_1^2 + 2b_{12}x_1x_2 + \cdots + b_{nn}x_n^2) \cos^2 \omega t,$$
$$T = \tfrac{1}{2}(a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + a_{nn}x_n^2)\omega^2 \sin^2 \omega t.$$

Since the system is conservative it is evident, from the last two equations, that the coefficients of $\cos^2 \omega t$ and $\sin^2 \omega t$ are equal. Equating these coefficients and solving for ω^2 we obtain

$$\omega^2 = \frac{b_{11}x_1^2 + 2b_{12}x_1x_2 + \cdots + b_{nn}x_n^2}{a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + a_{nn}x_n^2} = \frac{\overline{V}}{\overline{T}}. \qquad [99]$$

Obviously, ω^2 is a function of the amplitudes $x_1, x_2, \cdots, x_n$ of the motion. A necessary condition, from the calculus, for $\omega^2 = \dfrac{\overline{V}}{\overline{T}} = f(x_1, \cdots, x_n)$ to be maximum or minimum is that

$$d\omega^2 = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0, \qquad [100]$$

or

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0. \qquad [101]$$

When Eqs. (100–101) are satisfied then ω^2 is stationary. (See §1·6.) From (99–101)

$$\frac{\partial\omega^2}{\partial x_i} = \left[\overline{T}\frac{\partial\overline{V}}{\partial x_i} - \overline{V}\frac{\partial\overline{T}}{\partial x_i}\right]\frac{1}{\overline{T}^2} = 0,$$

or since $\overline{V} = \omega^2\overline{T}$

$$\frac{\partial\overline{V}}{\partial x_i} - \omega^2\frac{\partial\overline{T}}{\partial x_i} = 0 \quad (i = 1, 2, \cdots, n). \qquad [102]$$

Equations (102) when rearranged are

$$\begin{aligned} &(b_{11} - a_{11}\omega^2)x_1 + (b_{12} - a_{12}\omega^2)x_2 + \cdots + (b_{1n} - a_{1n}\omega^2)x_n = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \qquad [103] \\ &(b_{n1} - a_{n1}\omega^2)x_1 + (b_{n2} - a_{n2}\omega^2)x_2 + \cdots + (b_{nn} - a_{nn}\omega^2)x_n = 0. \end{aligned}$$

A necessary and sufficient condition that (103) possess a non-trivial solution for $x_1, x_2, \cdots, x_n$ is that the determinant Δ of the coefficients vanish, but this Δ is identical to that of §1·26. Thus $\Delta(\omega) = 0$ is the characteristic equation for Lagrange's equations and its roots $\omega_1, \omega_2, \cdots, \omega_n$ divided by 2π are the natural frequencies of the system.

Although the energy method displays no advantage, in determining the natural frequencies, over the method of §1·26 yet it furnishes the very important result that the amplitudes characteristic of the normal modes, i.e., the values of x_i satisfying (103) render $\omega^2 = \overline{V}/\overline{T}$ stationary.

1·37. Statement of Rayleigh's Principle. Rayleigh's principle is sometimes stated: *The distribution of the potential and kinetic energies, in the fundamental mode of vibration of an elastic system, is such as to render the frequency $\omega/2\pi$ a minimum.*

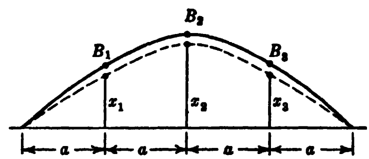

FIG. 1·39. Actual and Approximate Displacements in an Application of Rayleigh's Principle.

It may be of aid to interpret this principle with reference to a particular problem. Let the problem be to find the frequency of the fundamental mode of vibration of three equal masses attached to a light elastic string as shown in Fig. 1·39. The string is under tension S. If the system vibrates in its fundamental mode the form of the string is represented by the continuous line of the figure and B_1, B_2, and B_3 are the displacements characterizing the fundamental mode. It is supposed that the fundamental mode is *unknown*. Instead, it is known by observation that the mode resembles a parabola or a sine curve as represented by the broken curve. The displacements charac-

terizing these curves are x_1, x_2, and x_3 and these values specify a distribution of energy of the system. The quantities x_1, x_2, and x_3 also specify a constrained motion of the system. Applying the methods of energy we obtain the two formulas

$$\omega_1^2 = \frac{\bar{V}(B_1, B_2, B_3)}{\bar{T}(B_1, B_2, B_3)} \quad \text{and} \quad \omega_a^2 = \frac{\bar{V}(x_1, x_2, x_3)}{\bar{T}(x_1, x_2, x_3)}. \qquad [104]$$

The frequency $\omega_1/2\pi$ is the minimum frequency since B_1, B_2, and B_3 characterize the fundamental mode. The frequency $\omega_a/2\pi$ is a constrained frequency characterized by x_1, x_2, and x_3 and since the curve x_1, x_2, x_3 is almost the curve B_1, B_2, B_3 the values x_1, x_2, and x_3 will almost minimize ω_1^2. Rayleigh's principle states that $\omega_1 < \omega_a$. There are as many values of ω_a as there are curves resembling the continuous curve in Fig. 1·39. The only restrictions on x_1, x_2, x_3 are that they must satisfy a possible initial displacement of the system. Rayleigh's principle is important not only because $\omega_1 < \omega_a$ but *because ω_a is a good approximation to ω_1*.

EXAMPLE 1. Obtain approximate values for the fundamental pulsatance of the problem pertaining to Fig. 1·39. If q_1, q_2, q_3 are the coordinates of the system then

$$V = \frac{1}{2}\frac{S}{a}[q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2 + q_3^2],$$

$$T = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2).$$

If $q_i = x_i \cos \omega t$ is substituted in V and T the energy method yields

$$\omega_a^2 = \frac{S}{ma}\left[\frac{x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2}{x_1^2 + x_2^2 + x_3^2}\right].$$

If the three masses are estimated (guessed) to be on a parabola during the fundamental mode then $x_1 = x_3 = 3x_2/4$ and

$$\omega_a^2 = 0.5882\frac{S}{ma}.$$

If the three masses are estimated to lie on the sine curve $x = h \sin \pi t/(4a)$ then $x_1 = x_3 = \sqrt{2}\,h/2$ and $x_2 = h$ and

$$\omega_a^2 = 0.5970\frac{S}{ma}.$$

The exact value for ω_1^2 is

$$\omega_1^2 = 0.5858\frac{S}{ma}.$$

EXAMPLE 2. Two heavy discs, whose moments of inertia are I_1 and I_2 are supported vertically as indicated in Fig. 1·40. The constants of the mechanism are: $I_1 = 4$ slug-ft.^{2}, $I_2 = 6$ slug-ft.^{2}, $k_1 = 1$ lb. ft./radian, $k_2 = 2$ lb. ft./radian. Find the pulsatance of the fundamental mode of angular vibration.

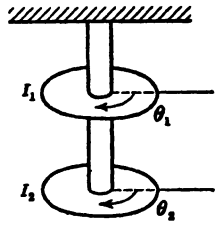

FIG. 1·40

The energies of the system are

$$V = \tfrac{1}{2}[k_1\theta_1^2 + k_2(\theta_2 - \theta_1)^2],$$

$$T = \tfrac{1}{2}[I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2].$$

If the system vibrates with frequency $\omega/2\pi$, i.e., according to the equation $\theta_i = x_i \sin \omega t$ then, by the energy method, ω will be a natural pulsatance for those values of x_1 and x_2 which render

$$\omega_a^2 = \frac{k_1x_1^2 + k_2(x_2 - x_1)^2}{I_1x_1^2 + I_2x_2^2} = \frac{x_1^2 + 2(x_2 - x_1)^2}{4x_1^2 + 6x_2^2}$$

stationary. If a fairly accurate estimate of the ratio of x_1 to x_2 in the fundamental mode of vibration can be made, these values will render ω^2 a minimum. By observation of the system it seems that $x_2 = 4x_1/3$. Substituting these values in ω_a^2 we obtain $\omega_a^2 = 1/12$. This is a good estimate since the exact value of $\omega_1^2 = 1/12$.

1·38. Proof of Rayleigh's Principle for Systems with a Finite Number of Degrees of Freedom. For systems with a finite number of degrees of freedom Rayleigh's principle is also stated: *The distribution of the potential and kinetic energies, in the fundamental mode of vibration of an elastic system, is such as to render the frequency a minimum and moreover the frequency of any simply periodic vibration lies between the greatest and least natural frequencies of the system.* The first part of this theorem, as stated in §1·37 has already been established in §1·36, i.e., the distribution of energies as represented by Eq. (99) is such as to render ω^2 a minimum for $\omega = \omega_1$ the fundamental pulsatance.

The second part of the theorem is best established by the use of normal coordinates. Of course the natural (normal) modes of vibration of an elastic system are independent of the coordinate system employed in its analysis and consequently the use of normal coordinates does not impair the generality of the second part of the proof.

It is recalled from §1·31 that in normal coordinates

$$V = \tfrac{1}{2}(b_1q_1^2 + \cdots + b_nq_n^2), \quad T = \tfrac{1}{2}(a_1\dot{q}_1^2 + \cdots + a_n\dot{q}_n^2),$$

and the solutions of Lagrange's equations are

$$q_i = A_i \sin \omega_i t \quad (i = 1, 2, \cdots, n),$$

where $\omega_i^2 = b_i/a_i$. A general pulsatance ω of a simply harmonic vibration, given by the energy method, is

$$\omega^2 = \frac{b_1x_1^2 + b_2x_2^2 + \cdots + b_nx_n^2}{a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2},$$

where the amplitudes $x_1, x_2, \cdots, x_n$ may or may not belong to a natural frequency.

Since $b_i = a_i\omega_i^2$ the last equation reduces to

$$\omega^2 = \frac{a_1\omega_1^2x_1^2 + a_2\omega_2^2x^2 + \cdots + a_n\omega_n^2x_n^2}{a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2}.$$

If ω_1 and ω_n are the least and greatest of the natural pulsatances $\omega_1, \omega_2, \cdots, \omega_n$ then it follows from the last equation that

$$\omega^2 - \omega_1^2 = \frac{a_2x_2^2(\omega_2^2 - \omega_1^2) + \cdots + a_nx_n^2(\omega_n^2 - \omega_1^2)}{a_1x_1^2 + \cdots + a_nx_n^2}$$

and

$$\omega_n^2 - \omega^2 = \frac{a_1x_1^2(\omega_n^2 - \omega_1^2) + \cdots + a_{n-1}x_{n-1}^2(\omega_n^2 - \omega_{n-1}^2)}{a_1x_1^2 + \cdots + a_nx_n^2}.$$

Since all the terms in parentheses in the last two equations are positive it follows that $\omega_1^2 < \omega^2 < \omega_n^2$.

1·39. Rayleigh's Principle and Continuous Systems. Rayleigh's principle as stated in §1·37 is true for continuously distributed systems.[16, 17]

EXAMPLE. Obtain, by means of Rayleigh's principle, approximations to the fundamental pulsatance of vibration of a uniform string of length l, linear density ρ, and under tension τ. The potential and kinetic energies are

$$V = \frac{\tau}{2}\int\left(\frac{\partial y}{\partial x}\right)^2 dx, \quad T = \frac{\rho}{2}\int\left(\frac{\partial y}{\partial t}\right)^2 dx.$$

If the manner of vibration is given by $y = z(x) \sin \omega t$ then, by the energy method,

$$\omega^2 = \frac{\overline{V}}{\overline{\overline{T}}} = \frac{\tau\int\left(\frac{\partial z}{\partial x}\right)^2 dx}{\rho\int z^2\, dx}.$$

[16] G. Temple and W. G. Bickley, *Rayleigh's Principle.*

[17] D. Prescott, *Applied Elasticity.*

(*a*) If the string is assumed to vibrate as a sine curve then $z = \sin \pi x/l$ and

$$\omega_1^2 = \frac{\tau\pi^2}{\rho l^2}\frac{\int_0^l \cos^2 \pi x/l\, dx}{\int_0^l \sin^2 \pi x/l\, dx} = \frac{\tau\pi^2}{\rho l^2} = \frac{9.87\tau}{l^2\rho}.$$

This is the exact value of the fundamental pulsatance given by the solution of the partial differential equation of the vibrating string.

(*b*) If the string is assumed to vibrate in the form of the parabola $z = (1 - 4x^2/l^2)$ then

$$\omega_a^2 = \frac{\overline{V}}{\overline{T}} = \frac{\tau\int_0^{l/2} 64x^2/l^4\, dx}{\rho\int_0^{l/2} (1 - 8\frac{x^2}{l^2} + 16\frac{x^4}{l^4})dx} = \frac{10\tau}{l^2\rho}.$$

(*c*) If the string is assumed to vibrate as two sides of a triangle, the equation of one side being $z = (1 - 2x/l)$ then

$$\omega_a^2 = \frac{\tau\int_0^{l/2} 4/l^2\, dx}{\rho\int_0^{l/2} (1 - 2x/l)^2\, dx} = \frac{12\tau}{l^2\rho}.$$

1·40. Orthogonality Condition. Rayleigh's principle gives the pulsatance of the fundamental mode. The second natural pulsatance can be found with but little additional labor by means of the so-called orthogonality relations. Let $x_1', x_2', \cdots, x_n'$ and $x_1'', x_2'', \cdots, x_n''$ denote the amplitudes characterizing respectively the fundamental and second smallest pulsatance of the elastic system. Then $x_1'', x_2'', \cdots, x_n''$ satisfy (103) and the equations

$$\sum_{i=1}^{n} x_i'' \frac{\partial \overline{V}}{\partial x_i'} = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i'' \frac{\partial \overline{T}}{\partial x_i'} = 0, \qquad [105]$$

where $\overline{V}$ and $\overline{T}$ are given by (99).

Equations (105) are established for a system of three degrees of freedom as follows. For this case $\overline{V}$ = constant and $\overline{T}$ = constant are equations of ellipsoids whose centers are at $x_1 = x_2 = x_3 = 0$. On any line $x_1 = c_1t$, $x_2 = c_2t$, $x_3 = c_3t$ (t a parameter), the ratio $\overline{V}/\overline{T}$ is a constant. In one particular direction this ratio is a minimum.

The equation of the plane tangent to $\omega^2 = \bar{V}/\bar{T}$ at the general point x_{10}, x_{20}, x_{30} is

$$\frac{\partial \omega^2}{\partial x_{10}}(x_1 - x_{10}) + \frac{\partial \omega^2}{\partial x_{20}}(x_2 - x_{20}) + \frac{\partial \omega^2}{\partial x_{30}}(x_3 - x_{30}) = 0.$$

It is recalled from analytic geometry that the partial derivatives are proportional to the direction cosines of the normal from the origin to the plane. If $\mathbf{r}'' = x''_{10}\mathbf{i} + x''_{20}\mathbf{j} + x''_{30}\mathbf{k}$ is *any* direction perpendicular to the normal to the plane then

$$(x''_{10}\mathbf{i} + x''_{20}\mathbf{j} + x''_{30}\mathbf{k}) \cdot \left(\frac{\partial \omega^2}{\partial x_{10}}\mathbf{i} + \frac{\partial \omega^2}{\partial x_{20}}\mathbf{j} + \frac{\partial \omega^2}{\partial x_{30}}\mathbf{k}\right) = 0$$

or, in view of the equation preceding Eq. (102)

$$\mathbf{r}''\cdot\left[\left(\frac{\partial \bar{V}}{\partial x_{10}} - \omega^2\frac{\partial \bar{T}}{\partial x_{10}}\right)\mathbf{i} + \left(\frac{\partial \bar{V}}{\partial x_{20}} - \omega^2\frac{\partial \bar{T}}{\partial x_{20}}\right)\mathbf{j} + \mathbf{k}\left(\frac{\partial \bar{V}}{\partial x_{30}} - \omega^2\frac{\partial \bar{T}}{\partial x_{30}}\right)\right] = 0.$$

The last equation is true for infinitely many values of ω^2. Consequently,

$$\sum_{i=1}^{3} x''_{i0}\frac{\partial \bar{V}}{\partial x_{i0}} = 0 \quad \text{and} \quad \sum_{i=1}^{3} x''_{i0}\frac{\partial \bar{T}}{\partial x_{i0}} = 0. \qquad [106]$$

Now x'_1, x'_2, x'_3 and x''_1, x''_2, x''_3 lie on perpendicular axes. Letting $x_{i0} = x'_i$ and $x''_{i0} = x''_i$ Eqs. (106) reduce to (105) for $n = 3$. Moreover, since for a natural frequency $\dfrac{\partial \bar{V}}{\partial x_i} = \omega_i^2\dfrac{\partial \bar{T}}{\partial x_i}$ (see Eq. 102), Eqs. (106) are dependent. Thus either the first or second of Eqs. (105) is the orthogonality condition.

EXAMPLE 1. Obtain the second lowest pulsatance of illustrative example 2, §1.37. Equations (105) for this example become

$$k_1x_1x'_1 + k_2(x_2 - x_1)(x'_2 - x'_1) = 0 \quad \text{and} \quad I_1x_1x'_1 + I_2x_2x'_2 = 0.$$

(The primes have been diminished by one.) By the energy method

$$\omega^2 = \frac{x_1'^2 + 2(x'_2 - x'_1)^2}{4x_1'^2 + 6x_2'^2}.$$

Substituting $x_2 = 4x_1/3$ in the second orthogonality relation we obtain $x'_2 = -x'_1/2$. When this relation is substituted in the expression for ω_2 we obtain $\omega_2 = 1$. This is the exact value for the second pulsatance.

EXAMPLE 2. Obtain ω_2 for the illustrative example of §1.39. Referring to the expression for $\bar{V}$ and $\bar{T}$ and (105) we have for the orthogonality conditions

$$\tau \int \frac{\partial z}{\partial x}\frac{\partial z'}{\partial x}\,dx = 0 \quad \text{and} \quad \rho \int zz'\,dx = 0.$$

The latter is

$$\rho \int (\sin \pi x/l)(z')dx = 0.$$

A value of z' satisfying this equation and the conditions of a possible initial displacement is $z' = \sin n\pi x/l$, $(n = 2, 4, \cdots)$. This value of z' is now to be used in the first expression for ω^2 in §1·39. Letting $n = 2$ and making this substitution, we have

$$\omega_2^2 = \frac{4\pi^2\tau}{\rho l^2}\,\frac{\int_0^l \cos^2 2\pi x/l\,dx}{\int_0^l \sin^2 2\pi x/l\,dx} = \frac{4\pi^2\tau}{l^2\rho}.$$

1·41. Summary. The procedure in the application of Rayleigh's principle is:

(*a*) Obtain expressions for the potential and kinetic energies of the system relative to its equilibrium position.

(*b*) If the system has a finite number of degrees of freedom substitute $q_i = x_i \sin \omega t$; if a continuous system let $y = z(x) \sin \omega t$.

(*c*) Solve for $\omega^2 = \bar{V}/\bar{T}$.

(*d*) Endeavor to minimize ω^2 by the substitution $x_i = c_i x$ or $z = z(x)$, where $c_i x$ or $z = z(x)$ characterizes either the fundamental mode of vibration or what is thought to be the fundamental. For this estimate of the fundamental mode the engineer is dependent upon knowledge of physical principles, intuition, experiment, and experience.

(*e*) If the system is one of a finite number of degrees of freedom the value of m obtained is an approximate or exact root of $\Delta(m) = 0$. (In general, it is easier to verify the solution of an algebraic or transcendental equation than to solve it.)

(*f*) The orthogonality conditions, leading to the second lowest frequency, are written by reference to Eqs. (105) or by analogy with the illustrative example of §1·40.

(*g*) If $\bar{T}$ and $\bar{V}$ denote the mean values of T and V taken over a cycle, the results of Sec. 7 are unchanged.

EXERCISES AND PROBLEMS XIV

1. Obtain, by Rayleigh's principle, approximations to lowest and second lowest frequencies of vibration of the double pendulum of Ex. 3, § 1·10, under the assumption that θ_1 and θ_2 are small.

2. Find, by Rayleigh's principle, an approximation to the fundamental frequency of vibration of the accelerometer of illustrative example 2, § 1·10.

3. Obtain, by Rayleigh's principle, an approximation to the fundamental frequency for the transverse vibration of a stretched uniform string having a mass M attached at the mid-point of the string. The mass per unit length of the string is ρ and the tension of the string is S. Show first that

$$\omega^2 = \frac{\bar{V}}{\bar{T}} = \frac{S\int\left(\frac{\partial z}{\partial x}\right)^2 dx}{\rho\int z^2\,dx + Mr_1^2}.$$

4. Show that the orthogonality conditions for Ex. 3 are

$$S\int\frac{\partial z}{\partial x}\frac{\partial z'}{\partial x}\,dx = 0 \quad\text{and}\quad \int\rho zz'\,dx + Mz_1z_1' = 0,$$

and obtain the second lowest frequency.

5. A revolving shaft is subject to transverse forces owing to its loading and impressed torque. When the shaft is deflected from its position at rest its motion consists of (*a*) revolution about its axis and (*b*) rotation about its undeflected axis at rest. The frequency of revolution depends upon the impressed torque. The frequency of rotation depends upon the distribution of kinetic and potential energies of the distorted shaft. If these frequencies coincide undesirable resonance exists. If the lateral displacement of the axis of the shaft is y at a distance x from one end then the potential energy [18] due to bending is

$$V_b = \tfrac{1}{2}EI\int_0^l\left(\frac{d^2y}{dx^2}\right)^2 dx,$$

where l is the length of the shaft, E is Young's modulus, and I is the moment of inertia of the area of a cross-section.

If an element of shaft has mass $m\,dx$ and its velocity of rotation is $2\pi\omega y$ then the kinetic energy of the shaft is

$$T = 2\pi^2 m\int_0^l \omega^2y^2\,dx.$$

Part of the bending of the shaft may be due to end thrust p. The shaft possesses potential energy due to this distortion, but it is not available for translation into kinetic energy. The expression for this energy is

$$V_p = \tfrac{1}{2}p\int_0^l\left(\frac{dy}{dx}\right)^2 dx.$$

The total potential energy is $V = V_b - V_p$.

[18] A. L. Kimball, *Vibration Prevention in Engineering.*

Obtain the frequency of the fundamental mode of lateral vibration (i. e., the fundamental rotational frequency) in case the shaft is mounted:

(*a*) in short bearings at both ends. [$y = EI\, d^2y/dx^2 = 0$ at the ends of the shaft $x = \pm l/2$. The origin is taken at a point midway between the bearings. Assume $y = c(l^2/4 - x^2)(5l^2/4 - x^2)$.]

(*b*) in long bearings at both ends. [$y = dy/dx = 0$ at $x = \pm l/2$, and $y = c(l^2/4 - x^2)^2$.]

(*c*) in one long and one short bearing at each end.

(*d*) in one long bearing at one end, other end free.

(8)

Additional Methods and References

A brief description of additional methods and a list of references to theory and applications follow.

1·42. **Equations of Appell and Béghin.** The equations of Appell are a generalization of the equations of Lagrange. The treatment of both holonomic and non-holonomic physical systems are reduced to a single system of equations of dynamics. (Ref. 8.) The equations of Béghin are an extension of the equations of both Lagrange and Appell. The extension is important with reference to service mechanisms ("aux mécanismes comportant un asservissement"), in particular to gyrostatic compasses of Anschütz and Sperry. (Ref. 9.)

1·43. **References.** Only a very limited number of names and references are given in this article because most of the papers and books cited contain bibliographies covering a portion of the field. References are arranged according to topics. In the final section of each chapter the elements of a reference to a paper are: name of author, title of paper, journal, series number [ ] if it exists, **volume**, page (year).

1. **Calculus of Variations.** For results in parameter representation and additional conditions for an extremum see G. A. Bliss, *Calculus of Variations*, Carus Mathematical Monographs, University of Chicago Press, 1925. Oscar Bolza, *Lectures on the Calculus of Variations*, University of Chicago Press, 1904; reprint G. E. Stechert and Company, New York, 1931. For Euler's equations and extremals of multiple integrals see A. R. Forsythe, *Calculus of Variations*, Cambridge University Press, London, 1927.

2. **Purely Theoretical Treatment of Dynamics.** G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society, New York, 1927.

3. **Limitations of Hamilton's Principle in Dynamics.** In the use of Hamilton's principle in the analysis of dynamical systems the contraints, if any, need not be independent of the time, but the contraints must not depend upon the velocities. See Paul E. Appell, *Mécanique Rationelle*, Gauthier-Villars, Paris, 1918.

4. **Gauss' Principle.** Gauss' principle in dynamics is applicable under still more general conditions than Hamilton's principle. W. D. MacMillan, *Statics and Dynamics of a Particle*, p. 419, McGraw-Hill Book Company, New York, 1927.

5. **Constraints in Dynamical Systems.** W. D. MacMillan, *op. cit.*, p. 306. N. W. Akímoff, *Elementary Course in Lagrange's Equations*, Chaps. I, II, III, Philadelphia Book Company, 1917. Horace Lamb, *Dynamics*, p. 301, Cambridge University Press, London, 1914.

6. Impulsive Motion. Lagrange's equations were modified by Lagrange for impulsive motion. E. T. Whittaker, *Analytical Dynamics*, Third Ed., p. 50, Cambridge University Press, 1927. J. H. Jeans, *Theoretical Mechanics*, p. 344, Ginn and Company, Boston, 1907.

7. Equations of Impact. E. T. Whittaker, *op. cit.*, p. 234.

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CHAPTER II

INTRODUCTION TO TENSOR ANALYSIS OF STATIONARY NETWORKS AND ROTATING ELECTRICAL MACHINERY

(1) Preliminary Description, (2) Matrices and Linear Transformations, (3) Preliminary Concepts of Tensor Analysis, (4) Stationary Networks; (a) General Theory, (b) All-Mesh Networks, (c) Mesh Networks, (d) Interconnection of Networks, (5) Non-mathematical Outline of the Nature of the Theory of Rotating Electrical Machinery, (6) Primitive Machine with Stationary Reference Axes, (7) Derived Machines with Stationary Reference Axes, (8) Primitive Machine with Rotating Reference Axes, (9) Derived Machines with Rotating Reference Axes, (10) Machines Under Acceleration, (11) Tensorial Method of Attack of Engineering Problems, (12) References.

This chapter is an introduction to methods of reducing electrical engineering problems to mathematical form by means of tensor analysis and the theories of Kron.

PART (A)

TENSOR ANALYSIS OF STATIONARY NETWORKS

Part A, consisting of Secs. (1-4), is concerned with the elementary theory of matrices, tensors, and the development of stationary network analysis.

(1)

Preliminary Description

This section is a brief non-mathematical description of the theories of the whole chapter. No mathematical knowledge is presupposed.

2·1. Historical Note on Tensors. Although the applications of tensors in engineering is of very recent date,[1] tensor analysis itself is by no means new. The study of tensors was begun by Christoffel in 1869 after the foundations of the subject were laid by Gauss and Riemann two or three decades earlier. The study was greatly advanced by Ricci and Levi-Civita in 1901 by the paper, "Méthodes de Calcul Differentiel Absolu." In 1916 Einstein called attention to the usefulness of the work of Ricci and Levi-Civita and since that date tensor analysis is often referred to as the "Mathematics of Relativity." The body of theory of tensor analysis is extensive and its applications in other branches of mathematics and physics are exceedingly numerous. Among the most important are the applications in differential geometry, calculus of variations, quantum mechanics, dynamics, elasticity, and thermodynamics.

2·2. Scope of Kron's Theories. The applications of Kron's theories are so numerous as to be bewildering. The methods of thought and analysis seem destined to extend to mechanical engineering as well. So many fields are already opened up that a generation may be required for their complete exploration. Some fields to which the methods have been applied are (*a*) all linear (stationary or moving) networks with lumped parameters, (*b*) every type of rotating electrical machine, (*c*) communication and transmission systems, (*d*) magnetic and electrostatic networks, (*e*) multi-electrode vacuum tube circuits, (*f*) interconnected systems of similar and dissimilar apparatus and machines, (*g*) generalizations of Maxwell's field equations, and (*h*) mechanical engineering [2] problems. It is not here feasible to catalogue exhaustively the multitudinous applications of the theory. It is preferable to obtain an impression of its partial scope and its various branches and their mutual relations from the outline of the table in Fig. 2·1.

2·3. Nature of the Theories. A *non-mathematical* description of the nature of selected portions of Kron's work may be of value before engaging in the detailed mathematical analysis of the theory.

Just as the theory of relativity is a physical theory distinct from tensor analysis and from any single or group of principles of advanced

[1] Gabriel Kron, *Tensor Analysis of Rotating Machinery*, I, 1932; II & III, 1933, mimeographed; "Non-Riemannian Dynamics of Rotating Electrical Machinery," *Journal of Mathematics and Physics*, 1934, pp. 103–194; "Analyse Tensorielle Appliquée à l'Art de l'Ingenieur," *Bulletin de l'Association des Ingenieurs Electriciens*, Liége, Belgium, Sept., Oct., 1936; Feb., 1937. (Prize paper of Fondation George Montefiore.)

[2] C. Concordia, "The Use of Tensors in Mechanical Engineering Problems," *General Electric Review*, July, 1936.

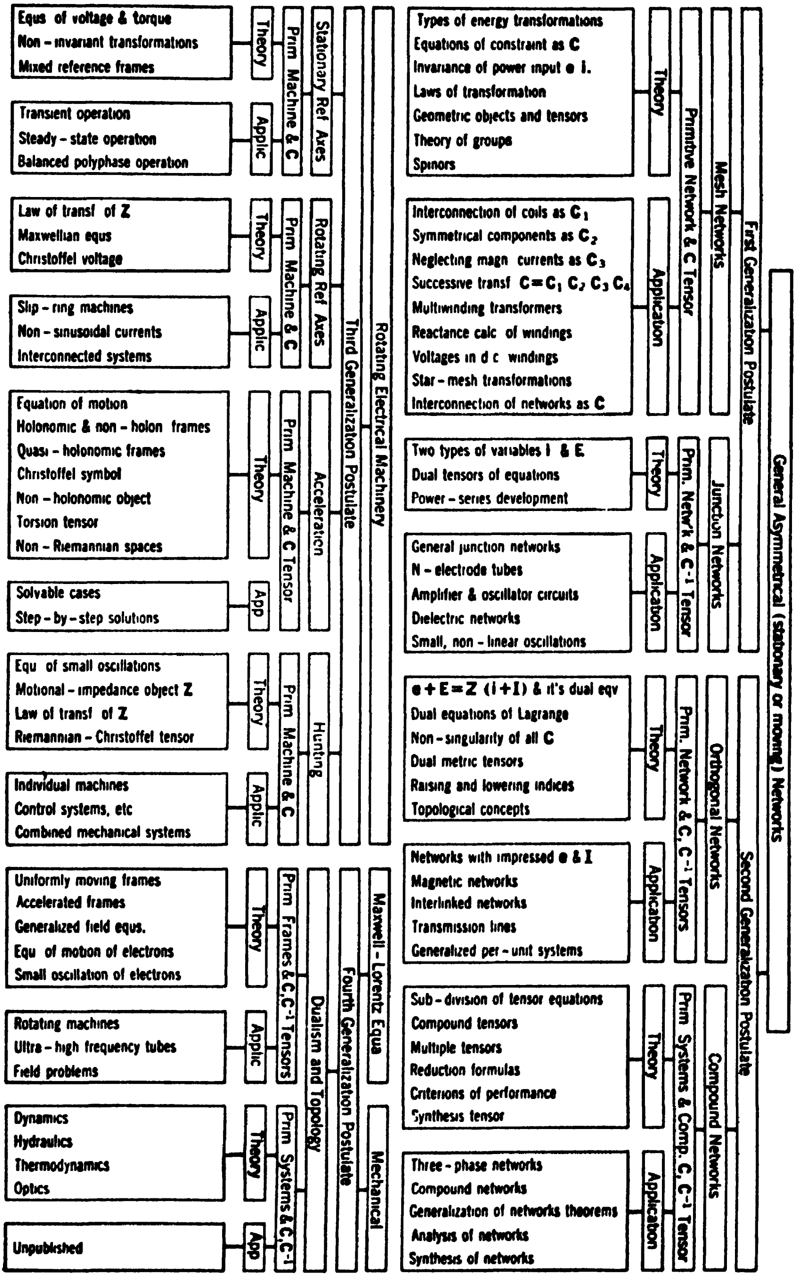
Equs of voltage & torque
Non-invariant transformations
Mixed reference frames
Transient operation
Steady-state operation
Balanced polyphase operation
Law of transf of Z
Maxwellian equs
Christoffel voltage
Slip-ring machines
Non-sinusoidal currents
Interconnected systems
Equation of motion
Holonomic & non-holon frames
Quasi-holonomic frames
Christoffel symbol
Non-holonomic object
Torsion tensor
Non-Riemannian spaces
Solvable cases
Step-by-step solutions
Equ of small oscillations
Motional-impedance object Z
Law of transf of Z
Riemannian-Christoffel tensor
Individual machines
Control systems, etc
Combined mechanical systems
Uniformly moving frames
Accelerated frames
Generalized field equs.
Equ of motion of electrons
Small oscillation of electrons
Rotating machines
Ultra-high frequency tubes
Field problems
Dynamics
Hydraulics
Thermodynamics
Optics
Unpublished
Theory
Applic
App
Prim Machine & C
Prim Machine & C Tensor
Prim Frames & C, C⁻¹ Tensors
Prim Systems & C, C⁻¹
Stationary Ref Axes
Rotating Ref Axes
Acceleration
Hunting
Dualism and Topology
Third Generalization Postulate
Fourth Generalization Postulate
Rotating Electrical Machinery
Maxwell-Lorentz Equa
Mechanical
Types of energy transformations
Equations of constraint as C
Invariance of power input e i.
Laws of transformation
Geometric objects and tensors
Theory of groups
Spinors
Interconnection of coils as C₁
Symmetrical components as C₂
Neglecting magn currents as C₃
Successive transf C=C₁ C₂ C₃ C₄
Multiwinding transformers
Reactance calc of windings
Voltages in d c windings
Star-mesh transformations
Interconnection of networks as C
Two types of variables i & E
Dual tensors of equations
Power-series development
General junction networks
N-electrode tubes
Amplifier & oscillator circuits
Dielectric networks
Small, non-linear oscillations
e+E=Z (i+I) & it's dual eqv
Dual equations of Lagrange
Non-singularity of all C
Dual metric tensors
Raising and lowering indices
Topological concepts
Networks with impressed e & I
Magnetic networks
Interlinked networks
Transmission lines
Generalized per-unit systems
Sub-division of tensor equations
Compound tensors
Multiple tensors
Reduction formulas
Criterions of performance
Synthesis tensor
Three-phase networks
Compound networks
Generalization of networks theorems
Analysis of networks
Synthesis of networks
Application
Primitive Network & C Tensor
Prim. Netw'k & C⁻¹ Tensor
Prim. Network & C, C⁻¹ Tensors
Prim Systems & Comp. C, C⁻¹ Tensor
Mesh Networks
Junction Networks
Orthogonal Networks
Compound Networks
First Generalization Postulate
Second Generalization Postulate
General Asymmetrical (stationary or moving) Networks

FIG. 2·1

physics so the epoch-making researches of Kron are much more than tensor analysis and advanced electrical engineering.

This achievement is such a discovery, generalization, and organization of those intrinsic physical entities common to a wide variety of similar and often seemingly totally dissimilar electrical and mechanical systems as to disclose the frequently multiple parallelism between the performances of these systems and to describe the behavior and relations between the entities of a system or systems by means of general mathematical equations whose forms are independent of the reference frames employed. The mathematical language of this work is tensor analysis.

Certain general features characterize this newest development.

(*a*) *Derivation.* It is *primarily* a systematic method of setting up and manipulating systems of differential and integral equations of performance of those problems in electrical and mechanical engineering which are expressible in terms of systems of differential and integral equations.[3]

(*b*) *Discovery and organization.* It has discovered and so organized the concepts of advanced electrical engineering that the derivation of the equations of performance of an unlimited number of physical systems and their general analysis and synthesis are reduced to routine manipulations and errors are largely precluded by extensive mathematical symmetry.

(*c*) *Power.* It is a method of great power. Its power consists in its:

(1) (Generalization) It unifies electrical engineering by substituting for a great multiplicity of separate and distinct theories of electrical devices certain broad general principles which supersede a patchwork of theories.

(2) (Routine operations) The quick reduction by routine methods of intricate problems to mathematical form which otherwise, if they can be reduced at all, are so reduced by the expenditure of great thought, waste of time, and toilsome effort.

(3) (Analogies) The generalization reveals analogies leading to the development of new machines and disclosing new relations between engineering and pure science.

(4) (Notation and generalized reference system) Throughout the analysis of any system only one equation of performance is required. The reference system is generalized in the sense that the equation of performance and other equations are valid without change for an infinite variety of coordinate systems. Thus, after the general equation has been derived for a simple coordinate system that special coordinate system can be selected which is most suitable for the solution of the problem at hand.

(5) (Modern algebraic theories) The analysis itself and the resulting equations of performance make available in engineering the power of modern algebra:

[3] If the system is linear and has constant coefficients the system is immediately solvable by tensor methods. It is solvable in numerous other cases.

matrices, group transformations, substitutions, elementary divisors, invariants, etc.

(6) (Modern analysis) The equations of performance are capable of physical interpretation and are of forms adapted to the methods of modern analysis and newly developed integrating machines.

It is impossible to give, in a few paragraphs, a clear, detailed, and comprehensive description of this achievement. It is, however, possible to sketch the construction and *modus operandi* of the new methods as restricted to the material of sections (3) and (4) of this chapter.

(*a*) *Stationary networks.* The differential equations of performance of a passive network of k meshes are

$$\sum_{j=1}^{k} z_{ij}^{(0)}(p)i_j^{(0)} = e_i^{(0)}, \quad (i, j = 1, 2, \cdots, k) \qquad [1]$$

where

$$z_{ij}^{(0)}(p)i_j^{(0)} = L_{ij}^{(0)}\, p\, i_j^{(0)} + R_{ij}^{(0)}\, i_j^{(0)} + \frac{1}{C_{ij}^{(0)}}\int i_j^{(0)}\, dt;$$

$i_1^{(0)}, i_2^{(0)}, \cdots, i_k^{(0)}$ are properly chosen mesh currents, and $e_1^{(0)}, e_2^{(0)}, \cdots, e_k^{(0)}$ are mesh voltages.

The differential equations of performance of the same network can also be written

$$\sum_{j=1}^{k} z_{ij}^{(1)}(p)i_j^{(1)} = e_i^{(1)}, \quad (i, j = 1, 2, \cdots, k) \qquad [2]$$

where

$$z_{ij}^{(1)}(p)i_j^{(1)} = L_{ij}^{(1)}pi_j^{(1)} + R_{ij}^{(1)}\, i_j^{(1)} + \frac{1}{C_{ij}^{(1)}}\int i_j^{(1)}\, dt;$$

$i_1^{(1)}, i_2^{(1)}, \cdots, i_k^{(1)}$ are k branch currents, not necessarily identical with $i_1^{(0)}, i_2^{(0)}, \cdots, i_k^{(0)}$.

Equations (1) and (2) are equations of performance of the same identical network. Equations (1) and (2) are similar in *form*, but are not identical. The quantities $(i_1^{(0)}, i_2^{(0)}, \cdots, i_k^{(0)})$ and $(i_1^{(1)}, i_2^{(1)}, \cdots, i_k^{(1)})$ are two different sets of dependent variables. This raises the question as to whether there is anything intrinsic or invariant regarding the network and its behavior, i.e., anything which remains unchanged under change of variables not only from $(i_1^{(0)}, i_2^{(0)}, \cdots, i_k^{(0)})$ to $(i_1^{(1)}, i_2^{(1)}, \cdots, i_k^{(1)})$, but under all possible changes of sets of variables.

The following questions are suggested: May there exist a hypo-

thetical current **i** (a vector quantity[4]) which can be expressed by many sets of components; one set of which is $(i_1^{(0)}, i_2^{(0)}, \cdots, i_k^{(0)})$, another $(i_1^{(1)}, i_2^{(1)}, \cdots, i_k^{(1)})$? May there exist a hypothetical voltage **e** (a vector quantity[4]) which can be expressed by *many* sets of components; one set of which is $(e_1^{(0)}, e_2^{(0)}, \cdots, e_k^{(0)})$, another $(e_1^{(1)}, e_2^{(1)}, \cdots, e_k^{(1)})$. May there exist a quantity **z**, labeled the impedance quantity of the network, which can be expressed in *many* sets of k^2 quantities; one set of which is $z_{ij}^{(0)}$ $(i, j = 1, 2, \cdots, k)$, another $z_{ij}^{(1)}$ $(i, j = 1, 2, \cdots, k)$? Does the scalar product $P = \mathbf{e}\cdot\mathbf{i}$ yield the power consumed in the circuit independently of the reference axes of **e** and **i**? The answer to all these questions is in the affirmative.

It is of course obvious that any network can be disconnected or broken up into n (n finite) distinct coils, where a coil is defined to be a

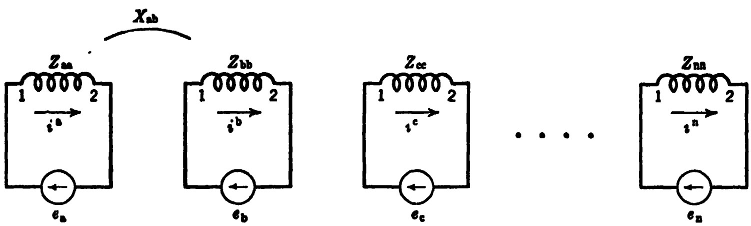

FIG. 2·2. Primitive Mesh Network.

portion of a circuit possessing an impedance which is independent of any component of **i** or **e**. We imagine any mesh network of physics or engineering so disconnected and the n component coils of the given network arranged in a linear configuration or sequence as in Fig. 2·2. The self-impedances of the n coils are denoted by z_{ii} $(i = 1, 2, \cdots, n)$. Whatever mutual impedances exist between the coils of the given network are indicated on the coils of the linear configuration. The mutual impedances are z_{ij} $(i, j = 1, 2, \cdots, n)$, $(i \neq j)$ which in the general case are asymmetrical. The n coils are each short-circuited. It is further supposed that there exists a voltage in series with each coil as indicated in Fig. 2·2. These n voltages are $\mathbf{e}_i$ $(i = 1, 2, \cdots, n)$. This linear sequence of n coils just described is called the **primitive network** for all-mesh network systems.

It is clear that a very large number of different prescribed stationary networks of engineering can be built by the proper connection of n coils, the only restriction on n being that it is finite. This raises the fundamental question of the entire matter: *Does there exist a mathemat-*

[4] The definition of vector in this case is *not* the definition of § 1·16, but is that given in § 2·17.

ical transformation or process, representable as a simple operator or symbol, which corresponds to the physical connection of n *coils into any prescribed mesh network and does the application of this process to the proper function of the parameters of the primitive network (and also, of course, parameters of the prescribed network) yield the differential equations of performance of the prescribed network into which the* n *coils of the primitive network are connected?* The answer is in the affirmative. The operator or symbol is Kron's connection tensor.

The full significance of this question and its answer must be understood. The primitive network for the given network and its differential equations are at once written down. The connection tensor is set up. The application of this tensor yields the differential equations of per-

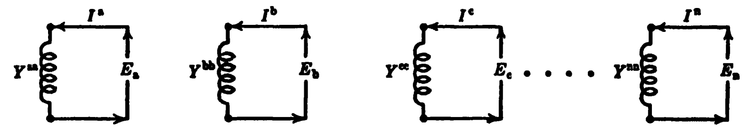

FIG. 2·3. Primitive Junction-pair Network.

formance of the given network. Three concepts stand out: (1) primitive network and its differential equations that are easy to establish; (2) interconnection of coils and its mathematical representation as a transformation; (3) given (or derived) network and its differential equations of performance which are to be found. The simplicity of the rules for these steps and the symmetry of the notation preclude the necessity of difficult thought and largely preclude the possibility of errors in algebraic sign or symbolism.

It is necessary, or at least convenient, to view certain given networks as junction-pair networks. This is true, for instance, with vacuum-tube and dielectric networks. Here the elements of the primitive network are not n short-circuited but n open-circuited coils. The variables in this case are not currents, but the n differences of potential existing across the n junction-pairs; the admittance tensor Y^{ij} is the dual of the impedance tensor Z_{ij} of mesh networks. The coils are arranged in a linear sequence as represented in Fig. 2·3. A connection tensor exists which is the dual of that of the mesh network and the procedure is similar to that for mesh networks.

The most general possible stationary networks are orthogonal networks. These are a combination of coils in which both currents and voltages are impressed. This generalization is effective also to provide a most general basis for the interconnection of electrical and mechanical networks into larger or super-networks.

Finally, it may be necessary to connect together, in an arbitrarily prescribed fashion, *m* such networks as described above. Each of the *m* networks may be separately analyzed. Next a connection tensor can be found which corresponds to the physical connection prescribed. The original work on the *m* individual networks is preserved. The final result is the differential equations of performance of the entire network, composed of the *m* smaller networks.

The theory finds its greatest usefulness when *hypothetical* reference frames (such as symmetrical components, magnetizing and load currents) and *hypothetical* design constants (bucking reactances) also are introduced. It should be mentioned that, in the analysis of a given network, just one equation of performance results. This is a tensor equation and it is the equivalent of a system of *n* differential equations. Thus the results are sometimes referred to as the equation or equations of performance.

(*b*) *Rotating electrical machinery.* The objective in this theory may be partially characterized by a comparison with the work of Lagrange in dynamics,[5] although the latter is comprised by the former in its broader aspects. Lagrange's equations, when adapted for constraints and holonomic and non-holonomic coordinates render the derivation of the system of differential equations of motion of dynamical systems largely a matter of routine. The equations of Lagrange formulate the dynamical problem as a system of equations to be solved; the method of Lagrange does not solve the differential equations. Kron's researches perform this same function for electrical networks, stationary or in motion; for vacuum-tube circuits; for every kind of rotating electrical machine under every kind of operating condition; and finally for all such systems interconnected. His work also establishes a routine procedure for the formulation of the equations of complicated physical systems by the aid of equations established first for simpler systems, the so-called primitive systems.

Hitherto a large portion of electrical engineering was given over to multitudinous diverse and independent theories of many machines. From previous points of view these machines were all different. Kron's work shows that all types of electrical rotating machines (whether direct current or alternating current) are mathematically identical except for interconnection of the windings and reference frames assumed. The primitive machine (there are in fact two, depending upon whether the reference axes are stationary or rotating), startling in its simplicity, is (discovered and) defined which includes, when the proper connections

[5] See Secs. 5 and 6, Chap. I.

have been made, all types of rotating machines as special cases. The primitive machine possesses all the fundamental physical and geometrical entities possessed by each individual type of machine, such as induction motor, compound direct-current motor, synchronous alternator, Schrage motor. The mathematical entity corresponding to the winding connections of the machine is the connection tensor. The application of the proper connection tensor (easily set up) with reference to the physical constants of the primitive machine produces, by a mathematical process no more difficult than matrix multiplication, the differential equations of performance of the specific machine under analysis. The analysis is complete for alternating-current or direct-current, symmetrical or asymmetrical machines under balanced or unbalanced loads, for steady-state or transient solutions, and with constant or accelerated rotor speed.

In addition to all this, the theory then passes on to the interconnection of rotating machines both with other machines and with other electrical and mechanical apparatus.

The work of Kron, because of its generalizations, power, synthesis of apparently diverse phenomena, symmetry and beauty of notation, and its interrelations with other branches of advanced mathematics and modern physics should be most pleasing to mathematicians and engineers.

The results and analysis of this new development are also expressible in the general language of multidimensional geometries.

2·4. Nature of the Present Approach. The approach to the operational calculus of Heaviside in Vol. I, Chap. IV, was a mathematical one, i.e., by means of ordinary linear differential equations and the theory of functions of a complex variable. This approach was rapid and necessitated no knowledge of engineering. It has been justified by the response from readers of the first volume.

The method of approach to the material of this chapter is likewise a mathematical one in the sense that the prerequisite pure mathematics employed is explained prior to entry upon the theories of the chapter. It is a mathematical approach also in the sense that the minimum engineering knowledge is presupposed.

(2)

Matrices and Linear Transformations

A knowledge of matrices and linear transformations is prerequisite for an introduction to tensor theory.

2·5. Definitions. The rectangular array

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ or } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ or } \begin{array}{|c|c|c|c|} \hline a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline \end{array}$$

composed of $m \times n$ numbers or functions a_{ij} is called a **matrix.** Abbreviated notations for the above matrix are **A** or (a_{ij}). The $m \times n$ numbers are the **elements** of the matrix. As a special case m may equal n. In this special case the matrix is not a determinant, although the matrix and the symbol of a determinant may be identical.

Two matrices, (a_{ij}) and (b_{ij}), each with m rows and n columns are **equal** only in case all corresponding elements are equal, i.e., $a_{ij} = b_{ij}$. A **zero matrix** is one, all of whose elements are zero. A **unit matrix** is a square matrix such that $a_{ij} = 1$, $i = j$ and $a_{ij} = 0$, $i \neq j$. A square matrix (a_{ij}) such that $a_{ij} = 0$, $i \neq j$ and $a_{ij} \neq 0$ for $i = j$ is called a **diagonal matrix.** A diagonal matrix each of whose diagonal terms is t is a **scalar matrix. The determinant** of the square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ is } \textit{the} \text{ determinant } \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}|.$$

The matrix (a_{ij}) is said to be **singular** or **non-singular** according as the determinant $|a_{ij}|$ does or does not vanish.

2·6. Rank, Adjoint, Transpose, Inverse, Sum. From the matrix (a_{ij}), possessing more than one element, other matrices may be formed by striking out of (a_{ij}) certain rows and columns. The determinants of the square matrices so formed, are called the determinants of (a_{ij}). A matrix is defined to be of rank r if there exists at least one r-rowed determinant of (a_{ij}) which is not zero while every determinant of order $(r + 1)$ of (a_{ij}) is zero.

The **adjoint** of the matrix **A** is defined as the matrix

$$\text{Adj. } \mathbf{A} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix}$$

where A_{ij} is the cofactor of the element a_{ij} in the determinant $|a_{ij}|$. The **cofactor** $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of the element a_{ij}. The minor M_{ij} is the $(n - 1)$ rowed determinant formed from $|a_{ij}|$ by deleting the ith row and jth column.

The matrix $\mathbf{A}_t$, formed from $\mathbf{A}$ by employing the successive rows of $\mathbf{A}$ as the successive columns of $\mathbf{A}_t$, is called the **transpose** of $\mathbf{A}$.

The **inverse** $\mathbf{A}^{-1}$ of the square matrix $\mathbf{A}$ is defined by the equation

$$\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{A_{11}}{a} & \cdots & \dfrac{A_{n1}}{a} \\ \cdots & \cdots & \cdots \\ \dfrac{A_{1n}}{a} & & \dfrac{A_{nn}}{a} \end{bmatrix}$$

where A_{ij} are the cofactors defined above and a is the determinant of (a_{ij}). The inverse of a non-square matrix is not defined.

A rule for rapid computation of $\mathbf{A}^{-1}$ is:

1. Write down the transpose $\mathbf{A}_t$ of $\mathbf{A}$.
2. Replace each element of $\mathbf{A}_t$ by its minor.
3. Divide each element of the matrix in (2) by the determinant a of $\mathbf{A}$.
4. Give to each element of the final matrix in (3) an algebraic sign according to the checkerboard array

$$\begin{matrix} + & - & + \cdots \\ - & + & - \cdots \\ + & - & + \cdots \\ \cdots & \cdots & \cdots \end{matrix}$$

The sum of two matrices (each $m \times n$) is defined to be an $m \times n$ matrix each of whose elements is the sum of the corresponding elements of the two given matrices. Likewise, the difference of two $m \times n$ matrices is an $m \times n$ matrix each of whose elements is the difference of the elements of the two original matrices. For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}.$$

EXERCISES I

1. Determine the rank of the matrix

$$\begin{bmatrix} 2 & 3 & 4 & 6 & 7 \\ 1 & 7 & -3 & \sin x & 4 \\ 1 & 2 & 4 & e & 3 \\ 4 & 6 & 8 & 12 & 14 \\ 2 & 3 & -1 & 7 & 2 \end{bmatrix}.$$

2. Compute the adjoint of each of the matrices

$$\begin{bmatrix} 2 & 3 & 5 & 4 \\ 1 & 3 & -1 & 2 \\ 4 & 6 & -5 & 3 \\ 1 & 7 & 9 & 11 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 2 & 0 & 3 \\ 1 & 4 & -1 & 2 \\ 3 & 4 & 6 & 1 \\ 7 & 5 & 2 & 3 \end{bmatrix}.$$

3. Compute the inverse of each of the above matrices.
4. Compute the inverse of the diagonal matrix

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix}.$$

5. A matrix **A** is symmetrical if $\mathbf{a}_{ij} = \mathbf{a}_{ji}$. A matrix **A** is said to be **skew-symmetric** if $a_{ij} = -a_{ji}, i \neq j$ and $a_{ii} = 0$. Show that a general matrix **A** can be expressed as **B** + **C** where **B** is a symmetric matrix and **C** is a skew-symmetric matrix.

2·7. Linear Forms, Linear Transformations. Matrices are important in the expansion of functions, the solution of systems of ordinary differential equations (Sec. 4, Chap. III), and in making transformations of variables. We begin with the simplest cases. The equations

$$\begin{aligned} L_1 &= a_{11}i_1 + a_{12}i_2 + \cdots + a_{1n}i_n, \\ L_2 &= a_{21}i_1 + a_{22}i_2 + \cdots + a_{2n}i_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ L_m &= a_{m1}i_1 + a_{m2}i_2 + \cdots + a_{mn}i_n, \end{aligned} \qquad [3]$$

define m linear forms in the n variables $i_1, \cdots, i_n$. Suppose that the n variables $i_1, \cdots, i_n$ are linearly expressible in terms of s new variables $i'_1, i'_2, \cdots, i'_s$, that is,

$$i_j = \sum_{k=1}^{s} b_{jk}\, i'_k \quad (j = 1, 2, \cdots, n). \qquad [4]$$

Substituting these values of i_j in Eqs. (3) we have

$$L_i = \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ij}\, b_{jk}\, i'_k = \sum_{k=1}^{s} c_{ik}\, i'_k \quad (i = 1, 2, \cdots, m) \qquad [5]$$

where

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} \quad (i = 1, 2, \cdots, m; \quad k = 1, 2, \cdots, s) \qquad [6]$$

$$= a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \cdots + a_{in}\, b_{nk}.$$

Since i ranges from 1 to m and k ranges from 1 to s the elements c_{ik} may be written out in the form of an m by s matrix $\mathbf{C}$ which is

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \quad + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \quad + a_{1n}b_{n2} & a_{11}b_{1s} + a_{12}b_{2s} + \ldots + a_{1n}b_{1n} \\ a_{21}b_{11} + a_{22}b_{21} + \quad + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \quad + a_{2n}b_{n2} & a_{21}b_{1s} + a_{22}b_{2s} + \ldots + a_{2n}b_{ns} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \ . + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \quad + a_{mn}b_{n2} & a_{m1}b_{1s} + a_{m2}b_{2s} + \ . + a_{mn}b_{ns} \end{bmatrix}.$$

It is evident that this expression can be obtained from the matrices **A** and **B** by a routine manipulation. By inspection of matrix **C** and the two matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}$$

it is evident that the element in the first row and column of **C** can be obtained from **A** and **B** by multiplying the successive elements of the first row of **A** by the successive elements of the first column of **B** and adding the resulting n products. Likewise, the element in the ith row and kth column of **C** is obtained by multiplying the successive elements of the ith row of **A** by the successive elements of the kth column of **B** and adding the n products.

The matrix **C** is defined to be the **product** of **A** and **B**. This product will be written $\mathbf{A}\cdot\mathbf{B}$.

Finally, we have the important theorem that a linear transformation (Eqs. 4) with matrix **B** replaces a set of linear forms (Eqs. 3) with matrix **A** by a set of linear forms (Eqs. 5) with matrix $\mathbf{A}\cdot\mathbf{B}$.

This theorem is as useful in change of reference axes both in linear network analysis and in the theory of vibrations as in the study of projective geometry.

EXERCISES II

1. If

$$\begin{aligned} L_1 &= 2x_2 + 3x_2 - 4x_3 + 7x_4, \\ L_2 &= -2x_1 + 7x_2 - x_3 + 3x_4, \\ L_3 &= 7x_1 + 5x_2 - 9x_3 + x_4, \\ L_4 &= 11x_1 + x_2 - x_3 + 3x_4, \end{aligned}$$

and

$$\begin{aligned} x_1 &= y_1 - y_2 + 2y_3 + y_4, \\ x_2 &= -y_1 - y_2 + 7y_3 + 3y_4, \\ x_3 &= 2y_1 - 3y_2 + 5y_3 + 7y_4, \\ x_4 &= 3y_1 + 5y_2 + y_3 + 5y_4, \end{aligned}$$

express, by means of the theorem of §2·7; L_1, L_2, L_3, L_4 as functions of y_1, y_2, y_3, y_4.

2·8. Multiplication of Matrices. The product $\mathbf{A}\cdot\mathbf{B}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ was defined in §2·7. If the positions of $\mathbf{A}$ and $\mathbf{B}$ are interchanged above and the multiplication indicated by the product $\mathbf{B}\cdot\mathbf{A}$ is performed, then it is evident that $\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$. Thus the multiplication of matrices is, in general, not commutative.

The multiplication of matrices is associative. Let $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$, $\mathbf{C} = (c_{ij})$ be any $m \times n$, $n \times s$, $s \times t$ matrices respectively. To see that multiplication is associative it is sufficient to show that a general element of $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$ is identical to the same element of $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$. By the rule for the product of two matrices (Eq. 6) the element il of $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$ is

$$\sum_{k=1}^{s}\left(\sum_{j=1}^{n} a_{ij}\, b_{jk}\right) c_{kl}. \tag{7}$$

The element jl of $\mathbf{B}\cdot\mathbf{C}$ is

$$\sum_{k=1}^{s} b_{jk}\, c_{kl},$$

and the element il of $\mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$ is

$$\sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{s} b_{jk}\, c_{kl}\right). \tag{8}$$

Since the finite sums (7) and (8) are identical $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$. The arrangements of the rows and columns, i.e., $m \times n$, $n \times s$, $s \times t$ should be noted. (See Ex. 4, problem set III.)

The reasoning above can be applied to any finite number of matrices, as long as the order of the matrices is preserved.

The multiplication of matrices is distributive. To show that $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ it is sufficient to show that a general element of the matrix $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})$ is equal to the corresponding element of the matrix $\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$. (The sum of matrices is defined in §2·6. The ik element of $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})$ is

$$\sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk}).$$

The ik element of $\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{C}$ is

$$\sum_{j=1}^{n} (a_{ij}\, b_{jk} + a_{ij}\, c_{jk}).$$

In the same manner it is shown that

$$(\mathbf{B}+\mathbf{C})\cdot\mathbf{A} = \mathbf{B}\cdot\mathbf{A} + \mathbf{C}\cdot\mathbf{A}$$

and the proofs are extensible to any number of matrices.

2·9. Division. Division by a non-singular (two [6]) matrix $\mathbf{A}$ is defined to be multiplication by $\mathbf{A}^{-1}$. The product of $\mathbf{A}^{-1}\cdot\mathbf{A}$ and $\mathbf{A}\cdot\mathbf{A}^{-1}$ is the unit matrix $\mathbf{I}$.

2·10. Differentiation and Integration. The derivative and integral of a matrix are defined as follows. The derivative with respect to a single variable t of a matrix is found by differentiating each component separately. For example,

$$\frac{d}{dt}\begin{bmatrix} t & 2 & \sin t \\ t^2 & t & \cos t \\ \sin t & 4 & \cos t \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos t \\ 2t & 1 & -\sin t \\ \cos t & 0 & -\sin t \end{bmatrix}.$$

The derivative of a matrix is, of course, a matrix.

A matrix is integrated with respect to a single variable by integrating each of its components separately. The integral of a matrix is a matrix.

EXERCISES III

1. Compute the product $\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 & -3 \\ -7 & 8 & 9 & 2 \\ 6 & 3 & 1 & 0 \\ 11 & 9 & 7 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 6 & -1 \\ 7 & 0 \\ 2 & 4 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 6 & 3 & 9 & 11 \\ 7 & 2 & 1 & -1 \end{bmatrix}.$$

[6] Division for more general matrices is not defined.

2. Verify that:

(a) $(\mathbf{A}\cdot\mathbf{B})_t = (\mathbf{B}_t)\cdot(\mathbf{A}_t)$,

(b) $(\mathbf{A}\cdot\mathbf{B})^{-1} = (\mathbf{B}^{-1})\cdot(\mathbf{A}^{-1})$,

(c) $(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C})_t = (\mathbf{C}_t)\cdot(\mathbf{B}_t)\cdot(\mathbf{A}_t)$,

(d) $(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C})^{-1} = (\mathbf{C}^{-1})\cdot(\mathbf{B}^{-1})\cdot(\mathbf{A}^{-1})$.

3. Prove that scalar matrices are the only matrices commutative with every $n \times n$ matrix.

4. The proof given in §2·8 for the validity of the associative law in matrix multiplication was for the three matrices **A**, **B**, and **C** of dimensions respectively $m \times n$, $n \times s$, $s \times t$. Show that

$$\mathbf{A}\cdot(\mathbf{e}\cdot\mathbf{B}) \neq (\mathbf{A}\cdot\mathbf{e})\cdot\mathbf{B}$$

if **A** and **B** are $n \times n$ and **e** is $l \times n$ matrix.

5. Prove that, for m a scalar,

$$m\mathbf{A} = \begin{bmatrix} ma_{11} & ma_{12} & \cdots & ma_{1n} \\ ma_{21} & ma_{22} & \cdots & ma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ma_{s1} & ma_{s2} & \cdots & ma_{sn} \end{bmatrix}.$$

6. The equation

$$\varphi(\lambda) = |\lambda I - [a]| = 0;$$

where $[a]$ is an n-rowed square matrix whose elements a_{ij} are constant, I is a unit matrix, λ is a parameter, and $|\lambda I - [a]|$ is a determinant, is called the characteristic equation of $[a]$. If $[a]$ is a square matrix and $\varphi(\lambda) = 0$ is its characteristic equation then $\varphi([a]) = 0$. Verify this theorem for $[a]$ a square matrix of order three.

7. Compute the derivative of the matrix

$$\begin{bmatrix} 1 & 2 & \sin x \\ 2 & 2e & \sin 2x \\ \sin x & \sin 2x & 3e \end{bmatrix}.$$

8. Compute the derivative of the *determinant*

$$\begin{vmatrix} 1 & 2 & \sin x \\ 2 & 2e & \sin 2x \\ \sin x & \sin 2x & 3e \end{vmatrix}.$$

9. Compute the reciprocal, in terms of the determinant, of the *determinant* itself in Ex. 8.

2·11. Three-matrices. The matrices of §§2·5–2·10 are 2- or 1-matrices, i.e., the number of elements in such matrices is $m \times n$ or $1 \times n$ and the elements are arranged in either a rectangle or line. In the applications which follow the elements of a matrix are parameters

belonging to some circuit or physical system. It is necessary to identify these elements with the physical system. For example, if **i** is the current written

$$\mathbf{i} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \cdots & \mathbf{k} \end{array} \\ \boxed{\begin{array}{c|c|c|c|c} i_a & i_b & i_c & \cdots & i_k \end{array}} \end{array}$$

then i_a may signify the component of **i** or the current associated with the a component or a mesh of the circuit. Likewise, if **z** is a matrix of impedances and is written

$$\mathbf{z} = \begin{array}{c|ccccc} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & z_{aa} & z_{ab} & z_{ac} & \cdots & z_{an} \\ \mathbf{b} & z_{ba} & z_{bb} & z_{bc} & \cdots & z_{bn} \\ \mathbf{c} & z_{ca} & z_{cb} & z_{cc} & \cdots & z_{cn} \\ \vdots & \cdots\cdot & \cdots\cdot & \cdots\cdot & \cdots\cdot & \cdots\cdot \\ \mathbf{m} & z_{ma} & z_{mb} & z_{mc} & & z_{mn} \end{array}$$

then z_{bc} may denote the mutual impedance between mesh b and mesh c.

A 3-matrix possesses $m \times n \times r$ elements. These may be arranged in a box. The symbol A_{abc}, for example, denotes a specific component

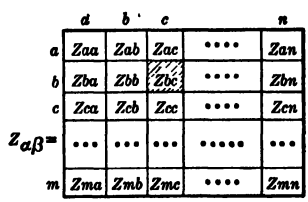

FIG. 2·4. Two-matrix.

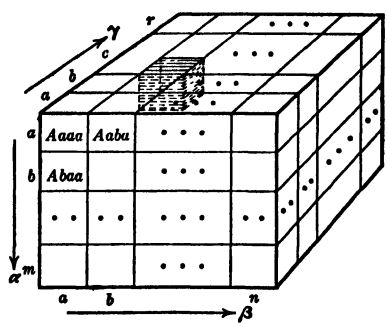

FIG. 2·5. Three-matrix.

or element of **A**, namely, that component located in the shaded volume of Fig. 2·5.

2·12. Index Notation. Evidently, for higher dimensional matrices a more convenient notation is imperative. Accordingly, the symbol

$A_{\alpha\beta}$ will denote a 2-matrix such as shown in Fig. 2·4. The Greek letters α and β are known as **variable indices**. It is understood that they assume all values $a, b, c, \cdots$. Thus $A_{\alpha\beta}$ denotes *all* the $m \times n$ components of a 2-matrix and these are arranged in a rectangle, i.e., the matrix itself. On the other hand, A_{bc} denotes one definite element of $A_{\alpha\beta}$, namely, the element shaded in Fig. 2·4. The Roman letters $a, b, c, \cdots$ are fixed indices. The symbol $A_{\alpha b}$ contains one variable and one fixed index. This symbol denotes the m elements of the b column of the matrix of Fig. 2·4.

The symbol $A_{\alpha\beta\gamma}$ denotes the 3-matrix of Fig. 2·5. The symbol $A_{\alpha\beta a}$ denotes the 2-matrix forming the face, nearest the reader, of the cube in Fig. 2·5. A 3-matrix can be represented on paper as a set of 2-matrices. For example, the 3-matrix of Fig. 2·5 can be represented as the set of r matrices

$$A_{\alpha\beta a},\ A_{\alpha\beta b},\ \cdots,\ A_{\alpha\beta r}.$$

EXERCISES IV

1. Represent, geometrically, i.e., by drawing rectangles and labeling with proper notations, the 4-matrix $A_{\alpha\beta\delta\gamma}$ ($\alpha = a, b, c, \cdots, m$; $\beta = a, b, c, \cdots, n$; $\delta = a, b, c, \cdots, r$; $\gamma = a, b, c, \cdots, s$) as a set of $(s \times r)$ 2-matrices.

2. Represent, geometrically, i.e., by drawing cubes and labeling with proper notations, the 5-matrix $A_{\alpha\beta\gamma\delta\epsilon}$ ($\alpha = a, b, c, \cdots, m$; $\beta = a, b, c, \cdots, n$; $\gamma = a, b, c, \cdots, r$; $\delta = a, b, c, \cdots, s$; $\epsilon = a, b, c, \cdots, t$) as a set of $(t \times s)$ 3-matrices.

3. Represent, geometrically, i.e., by drawing rectangles and labeling with proper notations, the 5-matrix of Ex. 2 as a set of $(t \times s \times r)$ 2-matrices.

2·13. Applications of Matrix Notation. (*a*) *Solution of linear, non-homogeneous, algebraic equations.* The system of equations

$$\begin{aligned} a_{11}\dot{i}_1 + a_{12}\dot{i}_2 + \cdots + a_{1n}\dot{i}_n &= e_1, \\ a_{21}\dot{i}_1 + a_{22}\dot{i}_2 + \cdots + a_{2n}\dot{i}_n &= e_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\dot{i}_1 + a_{n2}\dot{i}_2 + \cdots + a_{nn}\dot{i}_n &= e_n, \end{aligned} \tag{9}$$

can be written

$$\mathbf{A}\cdot\mathbf{i} = \mathbf{e} \tag{10}$$

where

$$\mathbf{A} = (a_{ij}),\ \mathbf{i} = \begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \cdot \\ \cdot \\ \dot{i}_n \end{bmatrix},\quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{bmatrix}.$$

Multiplying both sides of Eqs. (10) by $\mathbf{A}^{-1}$ we have

$$\mathbf{i} = \mathbf{A}^{-1}\cdot\mathbf{e},$$

which is the complete solution of Eqs. (9) for the variables $i_1, i_2, \cdots, i_n$.

(*b*) *Stationary circuit equations.* Equations (1) can be written

$$\mathbf{Z}\cdot\mathbf{i} = \mathbf{e}. \tag{11}$$

This equation is of the form of Eq. (10).

(*c*) *Differential equations of vibrations.* The potential and kinetic energies of a discrete dynamical system are given by Eqs. (54–55) Chap. I. Lagrange's equations (§1·12) for such a system are

$$\sum_{s=1}^{n}(a_{rs}\,\ddot{q}_s + b_{rs}\,q_s) = 0 \quad (r = 1, 2, \cdots, n). \tag{12}$$

These equations are written as the single matrix equation

$$[a][\ddot{q}] + [b][q] = 0. \tag{12a}$$

EXERCISES V

1. Solve, by the method of this article, the equations

$$\begin{aligned} i_1 + 2i_2 + 3i_3 + 4i_4 &= 34,\\ -i_1 + 3i_2 + 7i_3 + 2i_4 &= 36,\\ 4i_1 + 8i_2 + 5i_3 + 6i_4 &= 69,\\ 4i_1 + 7i_2 + 3i_3 + 4i_4 &= 39. \end{aligned}$$

2. Express Eq. (49) Chap. I in the form of Eqs. (12) of this chapter.

2·14. Solutions by Matrices. Sec. 4, Chap. III is devoted to the solution of systems of linear differential equations by means of matrices. Further theory of matrices such as raising of matrices to high powers and the expansion of analytic real functions in matrix form is found in the above section.

The solution, by Cramer's rule, of many linear equations in equally many unknowns is often cumbersome. A method of Kron, employing compound tensors,[7] reduces both the complexity and the probability of errors.

[7] Gabriel Kron, *Tensor Analysis of Networks*, Chap. IX. John Wiley and Sons, 1939.

(3)

Preliminary Concepts of Tensor Analysis

We shall need in Sec. 4 a preliminary knowledge of: (*a*) Definitions of tensors, (*b*) algebra of tensors, (*c*) inner multiplication and contraction, and (*d*) quotient law of tensors.

2·15. The Summation Convention. In sums, such as

$$\sum_{\alpha=1}^{n} e_\alpha i_\alpha, \quad \sum_{\beta=1}^{n} Z_{\alpha\beta} i_\beta, \quad \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} \Gamma_{\alpha\beta\gamma} i^\beta i^\gamma,$$

the summation signs may be omitted and the expressions written

$$e_\alpha i_\alpha, \quad Z_{\alpha\beta} i_\beta, \quad \Gamma_{\alpha\beta\gamma} i^\beta i^\gamma$$

if it is understood that whenever an index (subscript or superscript) appears twice in a term that term is to be summed for certain definite values (usually n) of the index. Thus

$$e_\alpha i_\alpha = e_1 i_1 + e_2 i_2 + \cdots + e_n i_n,$$

$$Z_{\alpha\beta} i_\beta = Z_{\alpha 1} i_1 + Z_{\alpha 2} i_2 + \cdots + Z_{\alpha n} i_n.$$

The index which appears twice is called a **dummy index.** A dummy index may be denoted by a different letter even in the same equation, e.g., $Z_{\alpha\beta} i_\beta = Z_{\alpha\gamma} i_\gamma$. The remaining indices of the equation are called **free indices.** In $Z_{\alpha\beta} i_\beta$, β is the **dummy index** and α is a **free index.**

The convention is not a mere abbreviation, but a valuable tool indicating and performing automatically certain operations in the analysis.

2·16. Definitions. Matrices of dimensions 1, 2, $\cdots$, n have been defined in Sec. 2 of this chapter. Suppose henceforth that the elements of the matrices are functions of general independent coordinates[8] $x^1, x^2, \cdots, x^n$. Let there be a change from the coordinate system $x^1, x^2, \cdots, x^n$ to the system $x^{1'}, x^{2'}, \cdots, x^{n'}$, where the $x^{i'}$ are defined by the equations

$$x^{i'} = \varphi^i(x^1, x^2, \cdots, x^n) \quad (i = 1, 2, \cdots, n) \tag{13}$$

and where the n functions φ^i are independent real functions of x^1, x^2, $\cdots$, x^n. Equations (13) define a transformation of coordinates of an

[8] The superscript n in x^n is not to be confused with an exponent. The symbol x^n is an abbreviation for $x^{(n)}$.

n-dimensional space. Since the n functions φ^i are independent, the x's are expressible in terms of $x^{i'}$'s as

$$x^i = \psi^i(x^{1'}, x^{2'}, \cdots, x^{n'}) \quad (i = 1, 2, \cdots, n). \qquad [14]$$

A tensor is sometimes defined as a matrix **A** plus a definite law of transformation for the components of the matrix when the coordinate system or reference frame is subjected to very general transformations. We shall see presently these definite laws of transformation whereby the components A^α, of the matrix **A** are changed to the components $A^{\alpha'}$ of a new matrix when the independent variables are changed from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ by means of Eqs. (14). In the following, the change of coordinates from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ is understood to be by the general transformation of Eqs. (14).

2·17. Tensors of Valence One. If under change of coordinates from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ the components of the 1-matrix A^m $(m = 1, 2, \cdots, n)$ are changed from A^m to $A^{m'}$ according to the law [9]

$$A^{m'} = \frac{\partial x^{m'}}{\partial x^m} A^m, \quad (m' = 1, 2, \cdots, n) \qquad [15]$$

then the 1-matrix is called a **contravariant tensor of valence one,** or a **contravariant vector.**

If under change of coordinates from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ the components of a 1-matrix A_m $(m = 1, 2, \cdots, n)$ are changed from A_m to $A_{m'}$ according to the law

$$A_{m'} = \frac{\partial x^m}{\partial x^{m'}} A_m, \quad (m = 1, 2, \cdots, n) \qquad [16]$$

then the 1-matrix is called a **covariant tensor of valence one,** or a **covariant vector.** The definitions of vectors just given are those referred to in §2·3. From these definitions it is to be noted that a vector does not necessarily possess either magnitude or direction. A vector is merely a matrix whose components obey, under change of reference system, one of the laws of transformation (15) or (16). Vectors of vector analysis possess physical significance, but their components are fictitious quantities. The vectors now considered are, in general, fictitious quantities but their components possess physical significance. However, the vectors of conventional vector analysis satisfy Eqs. (15) and (16) and occur as exceptional and special cases of the more general definitions of vectors here given. The restriction that a vector possess magnitude and direction is no longer imposed.

[9] For a clearer and more useful expression of this law see Sec. 4, §2·25.

EXAMPLE 1. The components of the vector **A** are A^1 and A^2 or $A^{1'}$ and $A^{2'}$ according as the reference axes are (x^1, x^2) or $(x^{1'}, x^{2'})$. (See Fig. 2·6*a*.) It is shown in this example that the new components $A^{1'}, A^{2'}$ of **A**, under a *particular* change of coordinates, are computed from the components A^1, A^2 of **A** by Eqs. (15). The equations of the

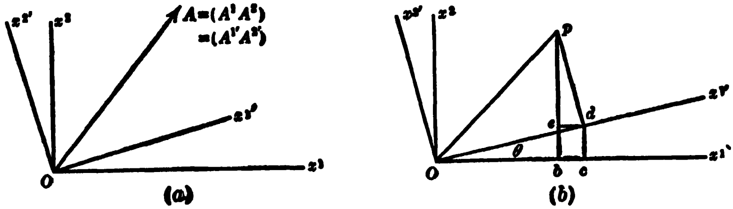

FIG. 2·6

particular transformation of coordinates under consideration are, from Fig. 2·6*b*,

$$x^1 = Oc - bc = x^{1'} \cos\theta - x^{2'} \sin\theta,$$

$$x^2 = cd + ep = x^{1'} \sin\theta + x^{2'} \cos\theta,$$

or

$$x^{1'} = x^1 \cos\theta + x^2 \sin\theta,$$

$$x^{2'} = -x^1 \sin\theta + x^2 \cos\theta.$$

From Eqs. (15)

$$A^{1'} = \frac{\partial x^{1'}}{\partial x^\beta} A^\beta = \frac{\partial x^{1'}}{\partial x^1} A^1 + \frac{\partial x^{1'}}{\partial x^2} A^2 = A^1 \cos\theta + A^2 \sin\theta,$$

$$A^{2'} = \frac{\partial x^{2'}}{\partial x^\beta} A^\beta = \frac{\partial x^{2'}}{\partial x^1} A^1 + \frac{\partial x^{2'}}{\partial x^2} A^2 = A^1(-\sin\theta) + A^2 \cos\theta.$$

By inspection of Fig. 2·6*b* it is evident that the components of $A^{m'}$ of the vector **A** in terms of the components A^m are given by the last equations.

If the new components of $A^{1'}, A^{2'}$ can be computed, by means of Eqs. (15), from A^1, A^2, not merely for the particular transformation

$$x^{1'} = x^1 \cos\theta + x^2 \sin\theta,$$

$$x^{2'} = -x^1 \sin\theta + x^2 \cos\theta,$$

but for all changes of coordinates $x^{i'} = \varphi^i(x^1, x^2, \cdots, x^n)$ possible relative to **A** then the vector **A** is a tensor of valence one.

EXAMPLE 2. The concept of covariant vector does not lend itself so easily to graphical illustration. Let $F(x^1, x^2, \cdots, x^n)$ be a scalar

function of position such that its value remains unchanged at a fixed point of space regardless of the coordinate system employed. Then the n quantities

$$\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}, \cdots, \frac{\partial F}{\partial x^n},$$

under transformation (14), are transformed according to Eqs. (16) and constitute the n components of a covariant vector or tensor of valence one. (See example 2 §2·18 for an illustration possessing more obvious physical significance.)

2·18. Tensors of Valence Two. If under change of coordinates from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ the components of a 2-matrix are changed from $Y^{\alpha\beta}$ to $Y^{\alpha'\beta'}$ according to the law

$$Y^{\alpha'\beta'} = Y^{mn}\frac{\partial x^{\alpha'}}{\partial x^m}\frac{\partial x^{\beta'}}{\partial x^n} \qquad [17]$$

then the set $Y^{\alpha\beta}$ is a **contravariant tensor of valence two.** The variables in $Y^{\alpha\beta}$ on the right side of Eqs. (17) are changed from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ by means of Eqs. (14).

If under change of coordinates from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ the components of a 2-matrix are changed from $Z_{\alpha\beta}$ to $Z_{\alpha'\beta'}$ according to the law

$$Z_{\alpha'\beta'} = Z_{mn}\frac{\partial x^m}{\partial x^{\alpha'}}\frac{\partial x^n}{\partial x^{\beta'}} \qquad [18]$$

then the set $Z_{\alpha\beta}$ is a **covariant tensor of valence two.**

EXAMPLE 1. *Covariant tensor, valence 2.* A rereading of the paragraph on stationary networks in §2·3 will be of aid in the following example.

Consider the circuit of Fig. 2·7. If i^1 and i^2 are the mesh currents of the network shown in Fig. 2·7*a*, then by Kirchhoff's second law the differential equations of performance are

$$\begin{aligned} Z_{11}i^1 + Z_{12}i^2 &= e_1, \\ Z_{21}i^1 + Z_{22}i^2 &= e_2, \end{aligned} \qquad \text{or} \quad Z_{mn}i_n = e_m \quad \text{or} \quad \mathbf{Z}\cdot\mathbf{i} = \mathbf{e}, \qquad [19]$$

where

$$Z_{11} = (L_{11} + L_{12})p + R_{11}, \quad Z_{12} = -L_{12}p,$$

$$Z_{21} = - L_{21}p, \qquad Z_{22} = (L_{22} + L_{12})p + R_{22} + \frac{1}{C_{22}p}.$$

Let us again write the differential equations of performance by using branch currents $i^{1'}$, $i^{2'}$ where, from Fig. 2·7*b*

$$\begin{aligned} i^1 &= i^{1'}, \\ i^2 &= i^{1'} - i^{2'} \end{aligned} \qquad \text{or} \quad \mathbf{i} = \mathbf{C}\cdot\mathbf{i}', \qquad [20]$$

where

$$\mathbf{i} = \begin{bmatrix} i^1 \\ i^2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{i}' = \begin{bmatrix} i^{1'} \\ i^{2'} \end{bmatrix}.$$

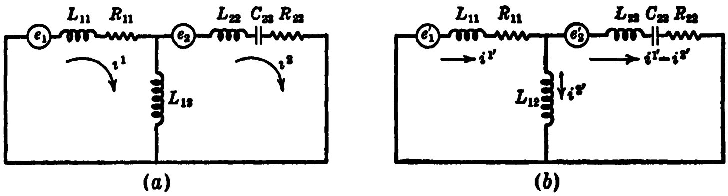

FIG. 2·7. Two Reference Frames.

The differential equations, by Kirchhoff's second law are

$$\begin{aligned} Z_{1'1'}i^{1'} + Z_{1'2'}i^{2'} &= e_{1'}, \\ Z_{2'1'}i^{1'} + Z_{2'2'}i^{2'} &= e_{2'} \end{aligned} \qquad [21]$$

where

$$\begin{aligned} Z_{1'1'} &= (L_{11} + L_{22})p + (R_{11} + R_{22}) + \frac{1}{C_{22}p}, \quad Z_{1'2'} = -\left(L_{22}p + R_{22} + \frac{1}{C_{22}p}\right), \\ Z_{2'1'} &= -\left(L_{22}p + R_{22} + \frac{1}{C_{22}p}\right), \quad Z_{2'2'} = (L_{22} + L_{12})p + R_{22} + \frac{1}{C_{22}p}, \end{aligned} \qquad [22]$$

and

$$e_1' = e_1 + e_2, \quad e_2' = -e_2.$$

Consider the matrices

$$\begin{bmatrix} Z_{11}i^1 & Z_{12}i^2 \\ Z_{21}i^1 & Z_{22}i^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Z_{1'1'}i^{1'} & Z_{1'2'}i^{2'} \\ Z_{2'1'}i^{1'} & Z_{2'2'}i^{2'} \end{bmatrix}.$$

Equations (20) are a special case of Eqs. (14). The quantities Z_{11} and $Z_{1'1'}$, for example, correspond to Z_{11} and $Z_{1'1'}$ of Eqs. (18). We

shall compute $Z_{1'1'}$ by means of Eqs. (18) and compare the values obtained with those of Eqs. (21–22). We have

$$\begin{aligned} Z_{1'1'} &= Z_{11}\frac{\partial i^1}{\partial i^{1'}}\frac{\partial i^1}{\partial i^{1'}} + Z_{12}\frac{\partial i^1}{\partial i^{1'}}\frac{\partial i^2}{\partial i^{1'}} \\ &\qquad + Z_{21}\frac{\partial i^2}{\partial i^{1'}}\frac{\partial i^1}{\partial i^{1'}} + Z_{22}\frac{\partial i^2}{\partial i^{1'}}\frac{\partial i^2}{\partial i^{1'}} \\ &= Z_{11} + Z_{12} + Z_{21} + Z_{22} \\ &= (L_{11} + L_{12})p + R_{11} - L_{12}p - L_{21}p \\ &\qquad + (L_{22} + L_{12})p + R_{22} + \frac{1}{C_{22}p} \\ &= (L_{11} + L_{22})p + R_{11} + R_{22} + \frac{1}{C_{22}p}. \end{aligned}$$

This is the same value for $Z_{1'1'}$ as that obtained from Eqs. (21–22).

By the same procedure it is easily shown that Z'_{21}, Z'_{12}, and Z'_{22}, for the particular equations of transformation given by Eqs. (20), are computed by Eqs. (18).

It must be pointed out that Eqs. (20) are only one special case of Eqs. (14) relative to Figs. 2·7. In Figs. 2·7 only two reference frames are shown. Many others exist. Twelve of these are shown in Fig. 2·8.

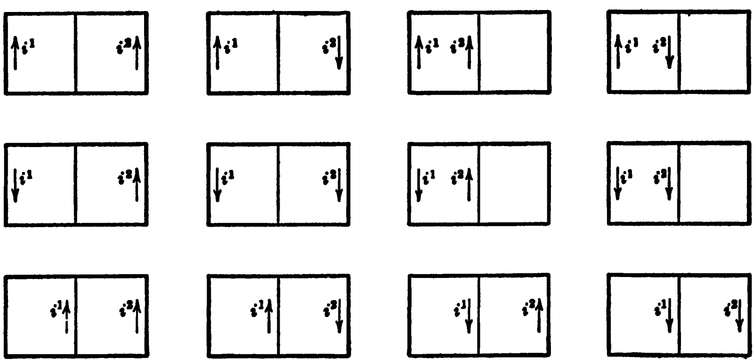

FIG. 2·8. Additional Reference Frames.

The remaining twelve are obtained by interchanging i^1 and i^2 on the diagrams of Fig. 2·8.

EXAMPLE 2. *Covariant tensor, valence one.* It is evident from Eqs. (21–22) that the 1-matrices $[e_1, e_2]$ and $[e_{1'}, e_{2'}]$ can be viewed as functions of i^1, i^2, $i^{1'}$, and $i^{2'}$. Let the components of $e_{1'}$ and $e_{2'}$ be

computed by Eqs. (16) where the equations of transformation of coordinates are Eqs. (20). We have

$$e_{1'} = \frac{\partial i^1}{\partial i^{1'}} e_1 + \frac{\partial i^2}{\partial i^{1'}} e_2 = e_1 + e_2,$$

$$e_{2'} = \frac{\partial i^1}{\partial i^{2'}} e_1 + \frac{\partial i^2}{\partial i^{2'}} e_2 = - e_2.$$

These values agree precisely with the values of $e_{1'}$ and $e_{2'}$ given by the last of Eqs. (22).

2·19. Mixed Tensors and Tensors of Higher Valence. If under change of coordinates from $x^1, x^2, \cdots, x^n$ to $x^{1'}, x^{2'}, \cdots, x^{n'}$ the components A_α^β of **A** are changed to $A_{\alpha'}^{\beta'}$ by the law

$$A_{\alpha'}^{\beta'} = \frac{\partial x^{\beta'}}{\partial x^n} \frac{\partial x^m}{\partial x^{\alpha'}} A_m^n, \tag{23}$$

then A_α^β is a mixed tensor of valence two.

The concept of tensor is generalized to those of higher valence. For example,

$$A_{\mu'\nu'\sigma'}^{\tau'} = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^\gamma}{\partial x^{\sigma'}} \frac{\partial x^{\tau'}}{\partial x^\delta} A_{\alpha\beta\gamma}^\delta \tag{24}$$

is the law of transformation for a tensor, contravariant of valence one and covariant of valence three.

2·20. Collection of Laws of Transformation. The classification of the formulas of §§ 2·17–2·19 are as follows:

Contravariant tensor, valence one $$A^{m'} = \frac{\partial x^{m'}}{\partial x^m} A^m \tag{25}$$

Covariant tensor, valence one $$A_{m'} = \frac{\partial x^m}{\partial x^{m'}} A_m \tag{26}$$

Contravariant tensor, valence two $$Y^{\alpha'\beta'} = Y^{mn} \frac{\partial x^{\alpha'}}{\partial x^m} \frac{\partial x^{\beta'}}{\partial x^n} \tag{27}$$

Covariant tensor, valence two $$Z_{\alpha'\beta'} = Z_{mn} \frac{\partial x^m}{\partial x^{\alpha'}} \frac{\partial x^n}{\partial x^{\beta'}} \tag{28}$$

Mixed tensor, valence two $$A_{\alpha'}^{\beta'} = A_m^n \frac{\partial x^{\beta'}}{\partial x^n} \frac{\partial x^m}{\partial x^{\alpha'}} \tag{29}$$

Mixed tensor, valence four $$A_{\mu'\nu'\sigma'}^{\tau'} = A_{\alpha\beta\gamma}^\delta \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^\gamma}{\partial x^{\sigma'}} \frac{\partial x^{\tau'}}{\partial x^\delta}. \tag{30}$$

In formulas (25–30) $x^1, x^2, \cdots, x^n$ are old coordinates, whereas $x^{1'}, x^{2'}, \cdots, x^{n'}$ always denote new coordinates. By inspection of the above formulas we have

When the indices are $\begin{cases} \textit{below} \\ \textit{above} \end{cases}$ *on the left side of the equation of transformation of components the new coordinates* $(x^{i'})$ *are* $\begin{cases} \textit{below} \\ \textit{above} \end{cases}$ *on the right side of the equation.*

EXERCISES VI

1. Write down the 24 matrices referred to in example 1, §2·18. Show that:

(*a*) If **A** and **B** are any two of the 24 matrices then **A**·**B** always yields a third matrix of the set.

(*b*) If **A**, **B**, and **D** are any three of the 24 matrices then $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{D} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{D})$.

(*c*) One **C** matrix is unit matrix, which belongs to the set.

(*d*) Each matrix of the set has an inverse which belongs to the set.

2. The components of the matrix A^{st}_{mnr} form a mixed tensor contravariant of valence 2 and covariant of valence 3. By analogy with the law of transformation of Eq. (30) write the law of transformation for the above tensor. If m, n, r, s, t each range over the integers 1, 2, 3, 4, how many components has the above tensor?

3. Express the rule for the multiplication of two 2-matrices in index notation.

4. A 3-matrix $A_{\alpha\beta\gamma}$ can be split up into 2-matrices $A_{\alpha\beta 1}, A_{\alpha\beta 2}, \cdots$, one for each fixed value of γ. The product of $A_{\alpha\beta\gamma}$ times $A_{\delta\epsilon}$ is a 3-matrix. This product can be formed by multiplying $A_{\alpha\beta 1}, A_{\alpha\beta 2}, \cdots$ sequentially by $A_{\delta\epsilon}$ and arranging the resulting product in a cube. Indicate these operations in index notation.

5. In example 1 of §2·17 let **A** be a vector (sense of conventional vector analysis) in 3-dimensional space. Let the transformation be from rectangular coordinates x, y, z to polar coordinates r, θ, φ, where the equations of transformation of coordinates are

$$\begin{aligned} x^1 &= x^{1'} \sin x^{2'} \cos x^{3'}, & x^{1'} &= [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}, \\ x^2 &= x^{1'} \sin x^{2'} \sin x^{3'}, & x^{2'} &= \tan^{-1}\left[\frac{(x^1)^2 + (x^2)^2}{x^3}\right]^{1/2}, \\ x^3 &= x^{1'} \cos x^{2'}, & x^{3'} &= \tan^{-1}\frac{x^2}{x^1}, \end{aligned}$$

where $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^{1'} = r$, $x^{2'} = \theta$, $x^{3'} = \varphi$. Obtain the components $A^{\alpha'}$ of **A** in the new system geometrically and by means of Eqs. (25) and establish their identity.

6. Compute, by means of transformation formula (28) the components $Z_{1'2'}, Z_{2'1'}, Z_{2'2'}$, ($Z_{1'1'}$ was computed in §2·18) for the circuit shown in Fig. 2·7.

7. Matrix **Y** is the inverse of matrix **Z**. Solve Eqs. (19) and (21) for **i** and **i′** obtaining $\mathbf{i} = \mathbf{Y}\cdot\mathbf{e}$ and $\mathbf{i}' = \mathbf{Y}'\cdot\mathbf{e}'$. Compute $Y^{1'1'}$ by means of the value Y^{11}, formula (27), and the equations of transformation $e_1' = e_1 + e_2$, $e_2' = -e_2$. Show that this value of $Y^{1'1'}$ is identical to that obtained in the equations $\mathbf{i}' = \mathbf{Y}'\cdot\mathbf{e}'$.

8. Compute, as in Ex. 7, $Y^{1'2'}$, $Y^{2'1'}$, and $Y^{2'2'}$.

2·21. Addition and Subtraction of Tensors. Two tensors are of the same **valence** if they possess the same number of indices. They are

of the same kind if they possess the same covariant and contravariant character.

The sum or difference of two tensors of the same kind is a tensor of the same kind. It will suffice to display the proof for two tensors each of valence four. Let the two tensors be

$$A^{\delta'\gamma'}_{\alpha'\beta'} = A^{n_1n_2}_{m_1m_2} \frac{\partial x^{\delta'}}{\partial x^{n_1}} \frac{\partial x^{\gamma'}}{\partial x^{n_2}} \frac{\partial x^{m_1}}{\partial x^{\alpha'}} \frac{\partial x^{m_2}}{\partial x^{\beta'}},$$

and

$$B^{\delta'\gamma'}_{\alpha'\beta'} = B^{n_1n_2}_{m_1m_2} \frac{\partial x^{\delta'}}{\partial x^{n_1}} \frac{\partial x^{\gamma'}}{\partial x^{n_2}} \frac{\partial x^{m_1}}{\partial x^{\alpha'}} \frac{\partial x^{m_2}}{\partial x^{\beta'}}.$$

Adding and subtracting these equations we have

$$A^{\delta'\gamma'}_{\alpha'\beta'} + B^{\delta'\gamma'}_{\alpha'\beta'} = (A^{n_1n_2}_{m_1m_2} + B^{n_1n_2}_{m_1m_2}) \frac{\partial x^{\delta'}}{\partial x^{n_1}} \frac{\partial x^{\gamma'}}{\partial x^{n_2}} \frac{\partial x^{m_1}}{\partial x^{\alpha'}} \frac{\partial x^{m_2}}{\partial x^{\beta'}}. \tag{31}$$

Since this equation is identical in form to the equation of transformation of each member of the sum, the sum itself is a tensor of valence four, covariant of valence two and contravariant of valence two.

2·22. Inner Product and Contraction. It is recalled from § 1·16 that the scalar product

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z,$$

where $\mathbf{A} = \mathbf{i}\,A_x + \mathbf{j}\,A_y + \mathbf{k}\,A_z$ and $\mathbf{B} = \mathbf{i}\,B_x + \mathbf{j}\,B_y + \mathbf{k}\,B_z$. It is pointed out that here the product of two vectors turns out to be a function of lower valence than either of the multipliers, i.e., turns out to be a scalar.

An analogous product exists in tensor analysis. In the last of Eqs (29) if $\beta' = \alpha'$ or $j' = i'$ then

$$A^{i'}_{j'} = A^k_l \frac{\partial x^{i'}}{\partial x^k} \frac{\partial x^l}{\partial x^{i'}} = A^k_k, \tag{32}$$

where k is a dummy index. This simplification is due to the relation

$$\frac{\partial x^{i'}}{\partial x^k} \frac{\partial x^l}{\partial x^{i'}} = \frac{\partial x^{1'}}{\partial x^k} \frac{\partial x^l}{\partial x^{1'}} + \frac{\partial x^{2'}}{\partial x^k} \frac{\partial x^l}{\partial x^{2'}} + \cdots + \frac{\partial x^{n'}}{\partial x^k} \frac{\partial x^l}{\partial x^{n'}}$$

$$= \frac{\partial x^l}{\partial x^k} = \begin{cases} 0 \text{ if } l \neq k \\ 1 \text{ if } l = k \end{cases} \tag{33}$$

since x^l and x^k are independent coordinates. If a function $A(x^{1'}, x^{2'}, \cdots, x^{n'})$ is obtainable from a function $A(x^1, x^2, \cdots, x^n)$ by means of the equations of transformation Eqs. (14) without employment of par-

tial derivatives, the A is a **scalar.** Evidently, in Eq. (32) A_k^k is a scalar. Since no derivatives appear in the transformation formula of a **scalar,** a scalar is also called a **tensor of valence zero.**

The product of two vectors represented in Eq. (32) is called the **inner product.** This product is called also **contraction** because a tensor of valence zero is obtained from a tensor A_j^i of valence two. The process of contraction is applicable not only to tensors of valence two, which are built as the product of two tensors of valence one, but to all mixed tensors of valence two. Moreover, it is applicable to all mixed tensors. Before establishing this fact it is convenient to introduce the concept of Kronecker deltas.

From Eqs. (13) the $x^{i'}$'s are functions of the x^i's and from Eqs. (14) the x^i's are functions of the $x^{i'}$'s. It is recalled that the $x^{i'}$'s and x^i's each form a set of n independent coordinates. Thus

$$\frac{\partial x^k}{\partial x^j} = \frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^j} = \delta_j^k = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

and [34]

$$\frac{\partial x^{k'}}{\partial x^{j'}} = \frac{\partial x^{k'}}{\partial x^i}\frac{\partial x^i}{\partial x^{j'}} = \delta_{j'}^{k'} = \begin{cases} 1 & \text{if } k' = j' \\ 0 & \text{if } k' \neq j'. \end{cases}$$

The deltas defined above are called **Kronecker deltas.**

In the mixed tensor A_{lm}^k the expression A_{lk}^k is the sum of n components of A_{lm}^k. That this sum is a tensor of valence one is shown as follows. From Eqs. (18)

$$\begin{aligned} A_{\alpha'\beta'}^{i'} &= A_{lm}^k \frac{\partial x^l}{\partial x^{\alpha'}}\frac{\partial x^m}{\partial x^{\beta'}}\frac{\partial x^{i'}}{\partial x^k} \\ &= A_{lm}^k \frac{\partial x^l}{\partial x^{\alpha'}}\frac{\partial x^m}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^k} \quad \text{if } \beta' = i' \\ &= A_{lm}^k \frac{\partial x^l}{\partial x^{\alpha'}}\delta_k^m = A_{lk}^k \frac{\partial x^l}{\partial x^{\alpha'}} \quad \text{for } m = k. \end{aligned}$$

In the two examples above it is evident that contraction has produced in each a tensor which is two valences lower than the original tensor. This result is true regardless of the valence of the original mixed tensor.

2·23. Quotient Rule. The physical dimensions of an unknown quantity occurring in a physical equation can be determined by the conditions that each term of the equation possess the same dimensions. In like manner the contravariant and covariant valence of an unknown

quantity entering a tensor equation can be inferred. For example, if

$$(A)B_{\alpha\beta} = C_{\gamma\beta}$$

then (A) must be a tensor of the form A^{α}_{γ}.

In fact there exists the theorem: A quantity which, when subjected to inner multiplication by an arbitrary covariant (or any arbitrary contravariant) vector, always yields a tensor is itself a tensor.

2·24. Summary. The results of §§2·15–2·23 are sufficient formal theory for the study of *stationary* networks. Additional formal parts of tensor analysis are developed as required in later sections.

An important aspect of tensor analysis is the fact that if a tensor equation holds in one system of coordinates, it continues to hold under any possible change of coordinate system. Example 1, §2·18 illustrates this important fact and anticipates, with a very simple and special example, the more general results of §2·40, Sec. 4.

New tensors are recognized by investigating directly their transformation laws, by the fact that the sum, difference, and product of two tensors is a tensor and by applications of the quotient law. Use is made of the second generalization postulate (§2·28) in establishing the tensor character of a mathematical entity.

EXERCISES VII

1. Show that if the Kronecker deltas are taken as the components of a mixed tensor of valence two in one set of coordinates then they are the components of a tensor in any set of coordinates.

2. Show that, by multiplication and contraction, the tensor $A_{ij}B^{jst}$ of valence three can be obtained from A_{ij} and B^{rst}.

3. Note that it is possible to have an inner product of two non-zero tensors equal to zero.

4. Represent on paper, by means of cube and rectangle, the processes carried out in example 2.

5. Show that, by multiplication and contraction, the scalar $A^{\alpha\beta}_{\alpha\beta}$ can be obtained from the tensors $A^{\alpha\beta}$ and $A_{\gamma\delta}$.

6. Show that the order of the factors in Exs. 2 and 5 is immaterial.

(4)

Stationary Networks

Sections 2 and 3 contain those elements of the classical theory of matrices and tensors prerequisite for the study of stationary networks.

Section 4 is an introduction to the study, by means of tensor analysis, of stationary networks.

(*a*)

GENERAL THEORY

2·25. Kron's Form of the Transformation Formulas. Kron expresses the transformation formulas (Eqs. 25–30) in a very convenient form. Let us re-examine formulas (25–30). Equations (13) and (14), namely,

$$x^{i'} = \varphi^i(x^1, x^2, \cdots, x^n) \quad \text{(New in terms of old variables)} \qquad [35]$$

$$x^i = \psi^i(x^{1'}, x^{2'}, \cdots, x^{n'}) \quad \text{(Old in terms of new variables)} \qquad [36]$$

with the conditions imposed in §2·16, remain the equations of transformation of variables. In Eqs. (25) and (26), if the indices m and α both range from 1 to n then both sets of quantities $\left[\dfrac{\partial x^{m'}}{\partial x^\alpha}\right]$ and $\left[\dfrac{\partial x^\alpha}{\partial x^{m'}}\right]$ may be defined in matrix form. The defining matrices are

$$\left[\frac{\partial x^{m'}}{\partial x^\alpha}\right] = \begin{bmatrix} \dfrac{\partial x^{1'}}{\partial x^1} & \dfrac{\partial x^{1'}}{\partial x^2} & \cdots & \dfrac{\partial x^{1'}}{\partial x^n} \\ \dfrac{\partial x^{2'}}{\partial x^1} & \dfrac{\partial x^{2'}}{\partial x^2} & \cdots & \dfrac{\partial x^{2'}}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x^{n'}}{\partial x^1} & \dfrac{\partial x^{n'}}{\partial x^2} & \cdots & \dfrac{\partial x^{n'}}{\partial x^n} \end{bmatrix} = C_\alpha^{m'} = \mathbf{C}^{-1} \qquad [37]$$

and

$$\left[\frac{\partial x^\alpha}{\partial x^{m'}}\right] = \begin{bmatrix} \dfrac{\partial x^1}{\partial x^{1'}} & \dfrac{\partial x^1}{\partial x^{2'}} & \cdots & \dfrac{\partial x^1}{\partial x^{n'}} \\ \dfrac{\partial x^2}{\partial x^{1'}} & \dfrac{\partial x^2}{\partial x^{2'}} & \cdots & \dfrac{\partial x^2}{\partial x^{n'}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x^n}{\partial x^{1'}} & \dfrac{\partial x^n}{\partial x^{2'}} & \cdots & \dfrac{\partial x^n}{\partial x^{n'}} \end{bmatrix} = C_{m'}^\alpha = \mathbf{C}. \qquad [38]$$

The matrix of Eq. (37) is denoted by $C_\alpha^{m'}$ (or $\mathbf{C}^{-1}$) and that of Eq. (38) by $C_{m'}^\alpha$ (or $\mathbf{C}$). The inverse of $\mathbf{C}$ is not, in general, calculated as the inverse of other matrices. (See Ex. 2.)

In view of the notations above, transformation formulas (15–16) become respectively

$$A^{m'} = A^{\alpha} C_{\alpha}^{m'}, \quad \text{or} \quad \mathbf{A}' = \mathbf{C}^{-1}\cdot\mathbf{A} \quad \text{(Components of } \mathbf{A} \text{ being contravariant)}, \qquad [39]$$

$$A_{m'} = A_{\alpha} C_{m'}^{\alpha}, \quad \text{or} \quad \mathbf{A}' = \mathbf{A}\cdot\mathbf{C} \quad \text{(Components of } \mathbf{A} \text{ being covariant)}, \qquad [40]$$

where the matrices $[A^{\alpha}]$ and $[A_{\alpha}]$ are components of contravariant and covariant vectors.

In the classical theory the attention is focused on Eq. (13). In this theory formulas (39–40) are the formulas for transformation of vector components. *In Kron's work the attention is directed on Eqs.* (14) *and the equivalent*[10] *of* (39–40) *which are*

$$A^{\alpha} = C_{m'}^{\alpha} A^{m'} \quad \text{or} \quad i^{\alpha} = C_{m'}^{\alpha} i^{m'} \quad \text{or} \quad \mathbf{i} = \mathbf{C}\cdot\mathbf{i}', \qquad [39a]$$

$$A_{\alpha} = C_{\alpha}^{m'} A_{m'} \quad \text{or} \quad e_{\alpha} = C_{\alpha}^{m'} e_{m'} \quad \text{or} \quad \mathbf{e} = \mathbf{e}'\cdot\mathbf{C}^{-1}, \qquad [40a]$$

where $A^{\alpha} = i^{\alpha}$ *and* $A_{\alpha} = e_{\alpha}$ *and* i^{α} *and* e_{α} *are usually components of current and voltage respectively. If there is a most fundamental equation, it is Eq.* (39*a*).

In the defining of Eqs. (37–38) for $\mathbf{C}^{-1}$ and $\mathbf{C}$ the arrangement of old and new variables is displayed in the partial derivatives themselves. It is much more convenient to display the reference frames or old and new variables by labeling the rows and columns of the tensors, i.e.,

$$\mathbf{C} = C_{m'}^{\alpha} = \begin{array}{c|c|c|c|} & \mathbf{a}' & \mathbf{b}' \cdots & \mathbf{n}' \\ \hline \mathbf{a} & C_{a'}^{a} & C_{b'}^{a} & C_{n'}^{a} \\ \hline \mathbf{b} & C_{a'}^{b} & C_{b'}^{b} & C_{n'}^{b} \\ \hline \vdots & \cdots & \cdots & \cdots \\ \hline \mathbf{n} & C_{a'}^{n} & C_{b'}^{n} & C_{n'}^{n} \\ \hline \end{array} \qquad [41]$$

$$A^{\alpha} = \begin{array}{|c|c|c|c|} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline A^{a} & A^{b} & \cdots & A^{n} \\ \hline \end{array} \qquad A_{\alpha} = \begin{array}{|c|c|c|c|} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline A_{a} & A_{b} & \cdots & A_{n} \\ \hline \end{array} \qquad [42]$$

[10] Attention must be given Eqs. (37, 38, 39, 40). The symbol $C_{\alpha'}^{\alpha}$ is a more general symbol than $\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}$. The variables in Eqs. (35, 36) denote holonomic coordinates. The corresponding electrical variables are charges. Since many electrical networks and machines are non-holonomic dynamical systems (See Sec. 6, Chap. I and §2·47.), in the general case no such relations can be established between the charges. However, in linear stationary networks, the $\mathbf{C}$ tensor can be found *formally* by $\frac{\partial i^{\alpha}}{\partial i^{\alpha'}}$, where the i and i' are currents (velocities). (For quasi-coordinates see Ref. 10 at end of Chap. I.)

The n Eqs. (39a) are now written as the single equation

$$\mathbf{i} = i^{\alpha} = \begin{array}{c|c|} \mathbf{a} & i^a \\ \mathbf{b} & i^b \\ \vdots & \vdots \\ \mathbf{n} & i^n \end{array} = \begin{array}{c|c|c|c|c|} & \mathbf{a}' & \mathbf{b}' & \cdots & \mathbf{n}' \\ \hline \mathbf{a} & C^a_{a'} & C^a_{b'} & \cdots & C^a_{n'} \\ \mathbf{b} & C^b_{a'} & C^b_{b'} & \cdots & C^b_{n'} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & C^n_{a'} & C^n_{b'} & \cdots & C^n_{n'} \end{array} \cdot \begin{array}{c|c|} \mathbf{a} & i^{a'} \\ \mathbf{b} & i^{b'} \\ \vdots & \vdots \\ \mathbf{n} & i^{n'} \end{array} \qquad [43]$$

The computation of the entire set of new components (new vector) is accomplished by the formal process of mere matrix multiplication. It is emphasized that the *formal manipulation* is matrix multiplication, but the analysis is not matrix analysis. The analysis is tensor analysis.

In the same manner formulas (27–30) are expressed

$$Y^{\alpha'\beta'} = Y^{mn}\ C^{\alpha'}_m C^{\beta'}_n, \quad \mathbf{Y}' = (\mathbf{C})^{-1}\cdot\mathbf{Y}\cdot(\mathbf{C}_t)^{-1} \qquad [44]$$

$$Z_{\alpha'\beta'} = Z_{mn}\ C^m_{\alpha'} C^n_{\beta'}, \quad \mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} \qquad [45]$$

$$A^{\alpha'}_{\beta'} = A^m_n\ C^{\alpha'}_m C^n_{\beta'} \qquad [46]$$

$$A^{\tau'}_{\mu'\nu'\sigma'} = A^{\delta}_{mns}\ C^m_{\mu'} C^n_{\nu'} C^s_{\sigma'} C^{\tau'}_{\delta}. \qquad [47]$$

The rule of §2·20 is transcribed to read: *When the indices are* $\left\{\begin{matrix}\textit{below}\\ \textit{above}\end{matrix}\right.$ *on the left side of the equation of transformation of components, the same indices are* $\left\{\begin{matrix}\textit{below}\\ \textit{above}\end{matrix}\right.$ *on the* **C**'*s. Whenever a dummy index appears* $\left\{\begin{matrix}\textit{below}\\ \textit{above}\end{matrix}\right.$ *on the components of the old tensor it appears* $\left\{\begin{matrix}\textit{above}\\ \textit{below}\end{matrix}\right.$ *on the* **C**'*s.*

The fundamental **C** to keep in mind is $\mathbf{C} = C^{\alpha}_{n'} = \left[\dfrac{\partial x^{\text{old}}}{\partial x^{\text{new}}}\right]$, which is given by Eq. (37). The fundamental equation of transformation to keep in mind is (39a) which corresponds to Eqs. (14) of §2·16. Equations (13) and (14) are the equations of transformation of the classical theory. The advantages of (39a) will appear in the applications which follow.

Equations (45) written explicitly are

$$\begin{array}{c|cccc|}
 & a' & b' & \cdots & n' \\
\hline
a' & Z_{a'a'} & Z_{a'b'} & \cdots & Z_{a'n'} \\
b' & Z_{b'a'} & Z_{b'b'} & \cdots & Z_{b'n'} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
n' & Z_{n'a'} & Z_{n'b'} & \cdots & Z_{n'n'} \\
\hline
\end{array}
=
\begin{array}{c|cccc|}
 & a & b & \cdots & n \\
\hline
a' & C^a_{a'} & C^b_{a'} & \cdots & C^n_{a'} \\
b' & C^a_{b'} & C^b_{b'} & \cdots & C^n_{b'} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
n' & C^a_{n'} & C^b_{n'} & \cdots & C^n_{n'} \\
\hline
\end{array}
\cdot$$

$$\begin{array}{c|cccc|}
 & a & b & \cdots & n \\
\hline
a & Z_{aa} & Z_{ab} & \cdots & Z_{an} \\
b & Z_{ba} & Z_{bb} & \cdots & Z_{bn} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
n & Z_{na} & Z_{nb} & \cdots & Z_{nn} \\
\hline
\end{array}
\cdot
\begin{array}{c|cccc|}
 & a' & b' & \cdots & n' \\
\hline
a & C^a_{a'} & C^a_{b'} & \cdots & C^a_{n'} \\
b & C^b_{a'} & C^b_{b'} & \cdots & C^b_{n'} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
n & C^n_{a'} & C^n_{b'} & \cdots & C^n_{n'} \\
\hline
\end{array}
\qquad [48]$$

EXERCISES VIII

1. Write one $C^a_{m'}$ for each of the illustrative examples of §§2·17–2·18, Sec. 3. Compute the corresponding $C^{m'}_a$.

2. If the equations of transformation are linear, $x^i = a_{ij}{}'x^{j'}$ $(i', j' = 1, 2, \cdots, n)$ show that $C^{m'}_a$ the inverse of $C^a_{m'}$ can be computed by the rule for computing the inverse of a matrix given in §2·6, Sec. 2.

2·26. Geometric Objects. A mathematical concept called a geometric object is now introduced which is more general than the concepts of matrix or tensor. Following Eqs. (22) appear the two matrices

$$\begin{array}{c|cc|}
 & i^1 & i^2 \\
\hline
i^1 & Z_{11} & Z_{12} \\
i^2 & Z_{21} & Z_{22} \\
\hline
\end{array}
\quad \text{and} \quad
\begin{array}{c|cc|}
 & i^{1'} & i^{2'} \\
\hline
i^{1'} & Z_{1'1'} & Z_{1'2'} \\
i^{2'} & Z_{2'1'} & Z_{2'2'} \\
\hline
\end{array}$$

The first results from a choice of Maxwell currents, while the second obtains from a choice of branch currents.

If reference frames different from the two introduced in Figs. 2·7 are chosen, additional matrices appear such as

$$\begin{array}{c|c|c|}
 & \mathbf{i}^{1''} & \mathbf{i}^{2''} \\
\hline
\mathbf{i}^{1''} & Z_{1''1''} & Z_{1''2''} \\
\hline
\mathbf{i}^{2''} & Z_{2''1''} & Z_{2''2''} \\
\hline
\end{array}$$

Each of these matrices is a different representation of a single underlying entity.

Analogous matrices exist for a general linear network. The totality of all possible matrices (one for each possible reference frame) indicate the existence of a quantity called a geometric object.

A **geometric object** is defined if:

(*a*) A particular n-matrix is given along one reference frame.

(*b*) All axes of this particular reference frame are specified.

(*c*) All possible reference frames are defined.

(*d*) The formula, that is, the "law of transformation" for finding the n-matrices along any possible reference frame is given.

Henceforth $Z_{(\alpha)(\beta)}$ (or any other letter, say $A_{(\alpha)(\beta)}$) will denote a 2-matrix. The symbol $Z_{\alpha\beta}$ will stand for a geometric object having components in a large or possibly infinite number of reference frames. For example, with reference to §2·18, example, $Z_{\alpha\beta}$ denotes the geometric object whose representations are 24 matrices.

In contrast then: (*a*) A matrix is an array of ordered components. (*b*) A tensor is a geometric object whose transformation formula, referred to in (*d*) above, is one of Eqs. (25–30). (*c*) A geometric object is characterized by the four defining specifications above and its transformation formula may be more general than those given in §2·25. The formula of transformation of a geometric object may be, but is not restricted to one of Eqs. (39, 40, 44–47). Thus a tensor is always a geometric object; a geometric object may, as a special case, be a tensor. In the study of stationary networks all geometric objects are tensors, but such is not true in the study of rotating machines.

Obviously, in the mathematical representation of a geometric object it is usually not possible to display all the matrix representations of the object. Instead there is given:

(*a*) An n-matrix showing the components of the geometric object in one specific reference frame.

(*b*) The specific reference frame is given.

(*c*) All possible reference frames are defined.

(*d*) The formula of transformation is expressed.

The number of dimensions of the n-matrix is the same as the valence of the geometric object.

In the example above Z_{mn} is a matrix showing the components of $Z_{\alpha\beta}$ in one specific reference frame such as

$$Z_{mn} = \begin{array}{c} \\ \mathbf{i}^1 \\ \mathbf{i}^2 \end{array}\begin{array}{cc} \mathbf{i}^1 & \mathbf{i}^2 \\ \hline \multicolumn{1}{|c|}{Z_{11}} & \multicolumn{1}{c|}{Z_{12}} \\ \hline \multicolumn{1}{|c|}{Z_{21}} & \multicolumn{1}{c|}{Z_{22}} \\ \hline \end{array} \quad \text{or} \quad Z_{mn} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \end{array}\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \hline \multicolumn{1}{|c|}{Z_{aa}} & \multicolumn{1}{c|}{Z_{ab}} \\ \hline \multicolumn{1}{|c|}{Z_{ba}} & \multicolumn{1}{c|}{Z_{bb}} \\ \hline \end{array}$$

A symbol Z_{aa}, say, represents one component of Z_{mn}.

2·27. First Generalization Postulate. In Sec. 2, Chap. I, Hamilton's principle was proved for dynamical systems. It can be established by separate individual proofs for certain other systems. There exists no single general proof establishing simultaneously its validity for all physical systems. In new situations it is *assumed* to hold and its validity checked by experiment. When used in this manner, Hamilton's principle is employed as a postulate.

In the remainder of this chapter two [11] important postulates are employed, which are called generalization postulates. The first generalization postulate as stated by Kron is: "*The method of analysis and the final equations describing the performance of a complex physical system (with n degrees of freedom) may be obtained by following step by step those of the simplest but most general unit of the system, provided each quantity is replaced by an appropriate n-dimensional matrix. The simplest unit of the system may have one or more degrees of freedom.*"

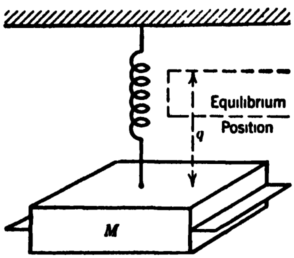

FIG. 2·9. One Degree of Freedom.

EXAMPLE 1. *Vibrating mechanical system.* As an illustration of the first postulate consider vibrating mechanical discrete systems. By inspection of Fig. 2·9 the differential equation of the motion of mass M is seen to be

$$a\ddot{q} + c\dot{q} + bq = 0, \qquad [49]$$

where a, b, and c are respectively the mass, spring, and damping constants and q is the displacement, at time t, of M from equilibrium position.

[11] In all, four exist.

Lagrange's equations of motion (§§1·12, 1·14, 1·25, Chap. I) of a vibrating dynamical system are

$$\frac{d}{dt}\frac{(\partial T)}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r} = - \frac{\partial V}{\partial q_r}, \quad (r = 1, 2, \cdots, n) \qquad [50]$$

where T and V are respectively the kinetic and potential energies and F is the Rayleigh dissipation function. The expressions for T, V, and F are

$$\begin{aligned} T &= \tfrac{1}{2}(a_{11}\, \dot{q}_1^2 + 2a_{12}\, \dot{q}_1\dot{q}_2 + \cdots + a_{nn}\, \dot{q}_n^2), \\ V &= \tfrac{1}{2}(b_{11}\, q_1^2 + 2b_{12}\, q_1q_2 + \cdots + b_{nn}\, q_n^2), \qquad [51] \\ F &= \tfrac{1}{2}(c_{11}\, \dot{q}_1^2 + 2c_{12}\, \dot{q}_1\dot{q}_2 + \cdots + c_{nn}\, \dot{q}_n^2). \end{aligned}$$

On substituting Eqs. (51) in Eqs. (50) Lagrange's equations can be written

$$\mathbf{a}\cdot\ddot{\mathbf{q}} + \mathbf{c}\cdot\dot{\mathbf{q}} + \mathbf{b}\cdot\mathbf{q} = 0 \quad \text{or} \quad a_{mn}\,\ddot{q}_n + c_{mn}\,\dot{q}_n + b_{mn}\,q_n = 0,$$

$$(m = 1, 2, \cdots, n) \qquad [52]$$

where

$$\mathbf{a} = \begin{bmatrix} a_{11}a_{12} \cdots a_{1n} \\ a_{21}a_{22} \cdots a_{2n} \\ \cdots\cdots\cdots \\ a_{n1}a_{n2} \cdots a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_{11}b_{12} \cdots b_{1n} \\ b_{21}b_{22} \cdots b_{2n} \\ \cdots\cdots\cdots \\ b_{n1}b_{n2} \cdots b_{nn} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_{11}c_{12} \cdots c_{1n} \\ c_{21}c_{22} \cdots c_{2n} \\ \cdots\cdots\cdots \\ c_{n1}c_{n2} \cdots c_{nn} \end{bmatrix}.$$

The system of Fig. 2·9 is characterized by the constants a, b, and c whereas the system of Eqs. (52) is characterized by the matrices **a**, **b**, and **c**. Equation (52) is obtained from Eq. (49) by replacing a, b, c, and q by **a**, **b**, **c**, and **q**.

EXERCISES IX

1. Reduce the differential equations of problems 5 and 7, problem set IX, Chap. I, to the form of Eqs. (52) of this chapter.

2. The partial differential equations for a single-wire (ground return) transmission line are

$$\frac{\partial e}{\partial x} = -Zi, \quad \frac{\partial i}{\partial x} = -Ye,$$

and

$$\frac{\partial^2 i}{\partial x^2} - m^2\, i = 0, \quad \frac{\partial^2 e}{\partial x^2} - m^2\, e = 0,$$

where

$$Z = R + Lp, \quad Y = G + Cp, \quad m^2 = ZY.$$

Consider a transmission *system* with n parallel conductors, electrostatic and electromagnetic coupling existing between the conductors. Arrange the resistances and

self- and mutual inductances in the matrix **Z** shown below. Likewise arrange the leakage conductances and self- and mutual capacitances in the matrix **Y**.

$$\mathbf{Z} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Z_{aa} & Z_{ab} & \cdots & Z_{an} \\ \mathbf{b} & Z_{ba} & Z_{bb} & \cdots & Z_{bn} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & Z_{na} & Z_{nb} & \cdots & Z_{nn} \\ \hline \end{array} \qquad \mathbf{Y} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Y_{aa} & Y_{ab} & \cdots & Y_{an} \\ \mathbf{b} & Y_{ba} & Y_{bb} & \cdots & Y_{bn} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & Y_{na} & Y_{nb} & \cdots & Y_{nn} \\ \hline \end{array}$$

The **i** and **e** matrices are

$$\mathbf{i} = \begin{array}{|cccc|} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline i^a & i^b & \cdots & i^n \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{|cccc|} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline e_a & e_b & \cdots & e_n \\ \hline \end{array}$$

(*a*) Show that the partial differential equations for the current and voltage on the n wire system are

$$\frac{\partial \mathbf{e}}{\partial x} = -\mathbf{Z}\cdot\mathbf{i} \quad \text{and} \quad \frac{\partial \mathbf{i}}{\partial x} = -\mathbf{Y}\cdot\mathbf{e}.$$

(*b*) By differentiation and substitution obtain the equations $\dfrac{\partial^2 \mathbf{e}}{\partial x^2} - \mathbf{m}^2\mathbf{.e} = 0$ and $\dfrac{\partial^2 \mathbf{i}}{\partial x^2} - \mathbf{m}^2\mathbf{.i} = 0$ from the equations $\dfrac{\partial \mathbf{e}}{\partial x} = -\mathbf{Z}\cdot\mathbf{i}$ and $\dfrac{\partial \mathbf{i}}{\partial x} = -\mathbf{Y}\cdot\mathbf{e}$.

(*c*) In a manner identical to that in (*b*) obtain the equations $\dfrac{\partial^2 \mathbf{e}}{\partial x^2} - \mathbf{Z}\cdot\mathbf{Y}\cdot\mathbf{e} = 0$ and $\dfrac{\partial^2 \mathbf{i}'}{\partial x^2} - \mathbf{Y}\cdot\mathbf{Z}\cdot\mathbf{i} = 0$ from the equations $\dfrac{\partial \mathbf{e}}{\partial x} = -\mathbf{Z}\cdot\mathbf{i}$ and $\dfrac{\partial \mathbf{i}}{\partial x} = -\mathbf{Y}\cdot\mathbf{e}$.

(The results in this problem are given by G. Kron in "The Application of Tensors to the Analysis of Rotating Electrical Machinery," *General Electric Review*, April, 1935.)

2·28. Second Generalization Postulate. The second postulate is a statement regarding the permanence of form of the tensor equations of physical systems. Its statement, as formulated by Kron, is: (*a*) "*The new system (under change of coordinates or reference frame) has the same number and types of n-matrices as the old system (namely,* **e**, **z**, *and* **i**) *but they now have different components.* (*b*) *The equation of the new system in terms of n-matrices is exactly the same as the equation of the old system, e.g.,* $\mathbf{e} = \mathbf{z}\cdot\mathbf{i}$. (*c*) *The n-matrices of the new system may be established from those of the old system by a routine transformation.*" That is, the matrix equation of a physical system is valid for an infinite number of analogous systems *of the same type* if each n-matrix is replaced by an appropriate geometric object having a permanent law

of transformation. The $\mathbf{C}$'s, transforming the various systems into each other, must be known.

EXAMPLE 1. *Network.* The differential equations of the two-mesh network of Fig. 2·7 is a very simple illustration of the second postulate. The equations, for two reference frames, are

$$\mathbf{Z}\cdot\mathbf{i} = \mathbf{e} \quad \text{and} \quad \mathbf{Z}'\cdot\mathbf{i}' = \mathbf{e}'.$$

The equations are identical in form. The matrices $\mathbf{Z}'$, $\mathbf{i}'$, and $\mathbf{e}'$ are established from $\mathbf{Z}$, $\mathbf{i}$, and $\mathbf{e}$ respectively by the routine transformation formulas of Eqs. (39, 40, 45).

EXAMPLE 2. *Dynamical system.* In example 1, §2·27, let Eqs. (51) be subject to a change of variable from $q_1, q_2, \cdots, q_n$ to $q_1', q_2', \cdots, q_n'$ by means of the equations of transformation

$$\begin{aligned} q_1 &= d_{11}q_1' + d_{12}q_2' + \cdots + d_{1n}q_n', \\ q_2 &= d_{21}q_1' + d_{22}q_2' + \cdots + d_{2n}q_n', \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_n &= d_{n1}q_1' + d_{n2}q_2' + \cdots + d_{nn}q_n', \end{aligned} \qquad [53]$$

whose matrix is $\mathbf{d}$.

Evidently, $V = \frac{1}{2}(b_{11}q_1^2 + 2b_{12}q_1q_2 + \cdots + b_{nn}q_n^2)$, although a quadratic form, can be viewed as a special case of a bilinear form. Accordingly, the bilinear form V with matrix $\mathbf{b}$ is replaced by a bilinear (quadratic) form V with matrix $\mathbf{d}_t\cdot\mathbf{b}\cdot\mathbf{d}$ when the q's are subject to the linear transformation (53).

The forms T and F can be transformed in a similar manner. Thus Eq. (52) becomes

$$\mathbf{a}'\cdot\ddot{\mathbf{q}}' + \mathbf{c}'\cdot\dot{\mathbf{q}}' + \mathbf{b}'\cdot\mathbf{q}' = 0, \qquad [54]$$

where

$$\mathbf{a}' = \mathbf{d}_t\cdot\mathbf{a}\cdot\mathbf{d}, \quad \mathbf{c}' = \mathbf{d}_t\cdot\mathbf{c}\cdot\mathbf{d}, \quad \mathbf{b}' = \mathbf{d}_t\cdot\mathbf{b}\cdot\mathbf{d} \qquad [55]$$

and $\mathbf{q}$ is the column matrix $(q_1, q_2, \cdots, q_n)$. Equations (52) and (54) are identical in form and Eqs. (55) furnish the routine formulas of transformation.

2·29. Stationary Network. The equation of performance of the network of Fig. 2·10 is

$$Z_{mn}(p)i^n = e_m \quad (m = 1, 2, \cdots, k), \qquad [56]$$

where $Z_{mn}(p)i^n = L_{mn}(p)i^n + R_{mn}i^n + \dfrac{1}{C_{mn}}\displaystyle\int i^n dt,$

$i^1, i^2, \cdots, i^k$ are k properly chosen mesh currents and $e_1, e_2, \cdots, e_k$ are mesh voltages.

If the network is extremely simple and k very small and the elements of the network merely wound coils, resistances, and capacitances the equation of performance can be easily written down by the direct application of Kirchhoff's laws. On the other hand if k is large, the network complex, and the elements are, in addition to the above elements, vacuum tubes, rotating machines, or if hypothetical currents

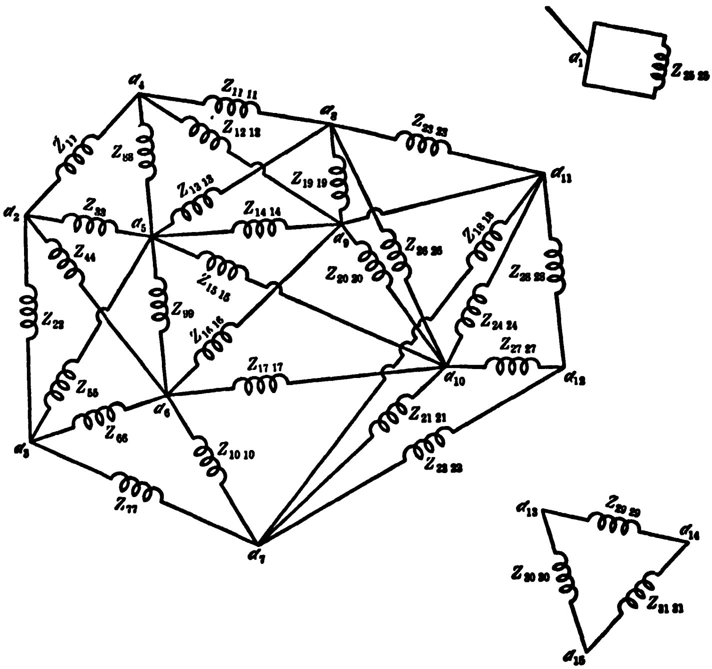

FIG. 2·10. Network, Sub-network, Junction-pair, Mesh.

(symmetrical components, magnetizing and load currents, etc.) are introduced, no such simple procedure will suffice. In such a simple network as that shown in Fig. 2·10 it may be difficult to determine even the minimum number of meshes or variables to be used.

2·30. Component Parts of Networks. In the networks considered the parameters are lumped. A network consists of two kinds of component parts: coils and junctions. No limitation is imposed on the physical nature of a coil. A coil may be a wound coil, a capacitance, vacuum tube, rotating machine winding, saturated reactor, etc. Electromagnetic and electrostatic couplings may exist between some or all the coils of the network. With each coil is associated certain numbers Z_{aa}, Y^{ab}, etc. No limitation is imposed upon the nature of the numbers

Z_{aa}, Y^{ab}, etc. These numbers associated with the coils (most frequently an impedance or admittance) may be real or complex numbers, functions of the time, or operators.

The two ends of a coil where it is joined to other coils are called **junctions.** When two or more junctions are joined with an impedanceless wire they are considered to be one junction. In Fig. 2·10, α_9 and α_{11} are one junction. The number of coils in the network of Fig. 2·10 is 31. There are 14 junctions.

A complete network may consist of a number of sub-networks. If between the pieces of a network there exist no physical connection' (electrical connection) then each such piece is called a sub-network. However, magnetic or dielectric couplings may exist between the sub-networks. The network of Fig. 2·10 consists of 3 sub-networks.

2·31. Analytical Units of Network. Any closed circuit in a network is called a **mesh.** The path $\alpha_3\alpha_6\alpha_7$ is a mesh. The path $\alpha_7\alpha_6\alpha_3$ is a mesh and it is the negative of the mesh $\alpha_3\alpha_6\alpha_7$.

Any two junctions located on the same sub-network are called a **junction-pair.** In Fig. 2·10 the junctions α_3, α_6 or α_3, α_9 constitute a junction-pair. The junction-pair $\alpha_6\alpha_3$ is the negative of the junction-pair $\alpha_3\alpha_6$.

Meshes and junction-pairs constitute the analytic units of a network. There exists an important theorem by which the least number of meshes, and the least number of junction-pairs required in the analysis of a network, are obtained by the mere process of counting.

2·32. Mesh, Junction-pair Theorem. From §§2·30–2·31 we have the concepts of coil, sub-network, junction; mesh and junction-pair. The number of each of these entities is not independent of the number of the others. Denote by S, J, M, P, and N the number of sub-networks, junctions, meshes, junction-pairs, and coils respectively of a given network. There exist the following theorems.[12]

I. The least number of junction-pairs of a network is equal to the number of junctions minus the number of sub-networks. In symbols

$$P = J - S. \tag{57}$$

II. The least number of meshes (required in the solution of a network) is equal to the number of coils minus the number of junction-pairs. In symbols

$$M = N - P. \tag{58}$$

Thus

$$M = N - (J - S) = N + S - J. \tag{59}$$

[12] O. Veblen: "Analysis Situs," *American Mathematical Society*, 1931, pp. 15–18.

The determination of the least number of meshes is reduced by Eq. (59) to the process of mere counting.

EXAMPLE. In the network of Fig. 2.10

$$M = 31 + 3 - 14 = 20 \text{ meshes.}$$

2·33. Types of Stationary Networks; Variables in Networks. It has been pointed out in §2·3 and also in Table I that the most general type of stationary networks, subject to the most general operating conditions, are orthogonal networks. However, a large class of electrical networks, subject to very general operating conditions, can be analyzed as mesh networks. The analyses of mesh, junction,[13] and orthogonal [14] networks are not unrelated. If the analysis of either of the first two is understood, a knowledge of the other is acquired without difficulty.

In example 1, §2·18, the variables of the simple network were taken first as mesh currents. Next branch currents were taken as variables. This is partially indicative of the choice of variables in complex networks. Either mesh currents, branch currents, or a combination of mesh and currents or hypothetical currents or differences of potential existing across coils of the network, or a combination of all of these quantities, may be taken as variables or coordinates in a network.

A network may be viewed as a configuration or arrangement of meshes. A network may be viewed also as a collection of junction-pairs. In the last case the variables are the differences of potential existing between the two junctions of the junction-pair.

In the most general network operating under the most general conditions a network must be viewed as both a collection of meshes and junction-pairs. The variables thence consist of the currents flowing in the meshes and the differences of potential existing across the junction-pairs.

Simple passive networks, multi-winding transformers, transmission lines, and rotating machines are primarily mesh networks. Multi-electrode vacuum tubes [15] are primarily junction-pair networks. A combination of vacuum-tube and transformer networks would produce a complete or orthogonal network.

2·34. Sign Conventions for Mutual Inductance. Two coils, with mutual inductance between them, can be connected in series in two different ways. If the connection is such that the flux coming from the

[13] Gabriel Kron, *Tensor Analysis of Networks*, Chap. XIV.

[14] *Ibid.*, Chap. XVI.

[15] Gabriel Kron, "Tensor Analysis of Multielectrode-Tube Circuits," *Electrical Engineering*, November, 1936. Also *Tensor Analysis of Networks*, Chap. XV.

first coil links the second in the same direction, then the connection is called **series aiding.** In going around a closed circuit in any direction the ends of the two coils are numbered 1-2 and 1-2 or 2-1 and 2-1 if the connection of the two coils is series aiding as shown in Fig. 2·11*a*.

If the connection is such that the flux coming from the first coil links the second in a direction opposite to the linkage in the first coil, then the connection is called **series opposing.** In going around a closed circuit in any direction the ends of the two coils are numbered 1-2 and

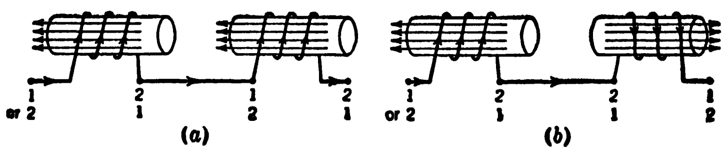

FIG. 2·11. Series-aiding and Series-opposing Connections.

2-1 or 2-1 and 1-2 if their connection is series opposing as shown in Fig. 2·11*b*. (See §2·39.)

(*b*)

ALL-MESH NETWORKS

An **all-mesh network** is one having the same number of coils as meshes.

2·35. Interconnection of Coils in All-mesh Networks. The network of Fig. 2·13*a*, §2·39, is an all-mesh network. Each coil of an all-mesh network is short-circuited upon itself, the ends of the coils being joined through a source of impressed voltage with the remainder of the network by means of impedanceless wires. With each coil is joined in series an impressed voltage. There are n coils, n impressed voltages, and n meshes. Some of the impressed voltages may be zero.

Imagine the all-mesh network broken up into n individual circuits, there being no electrical connections between the n simple circuits. Each simple circuit consists of a coil and an impressed voltage. (See Fig. 2·13*b*, §2·39.) A fact of great importance in the theory following is: The same currents flow through each of the n coils whether existing as a set of n individual coils or whether connected into an all-mesh network. The reason for this is that in either case the same voltage is in series with the coil and the circuit is then short-circuited upon itself by means of an impedanceless conductor. For example, the current through the coil Z_{aa} (Fig. 2·13*a*) is identical to the current through the coil Z_{aa} (Fig. 2·13*b*). Moreover, since the same value of current

flows, since the same voltage is impressed, and since the same impedance exists in both, the total instantaneous power input is identical.

All-mesh networks are not the practical networks of engineering. Practical networks, in general, contain more coils than meshes. Such networks are called mesh networks. However, the theory of the all-mesh network is first developed. This theory is then modified and extended so as to be applicable to mesh networks.

2·36. **Primitive System of Mesh Networks.** To establish the equation of performance of a given network the following procedure is used:

(*a*) Establish first the equations of another network, whose analysis is simple.

(*b*) Next change these equations into the equations of the given (hereafter called the derived) network by a routine transformation.

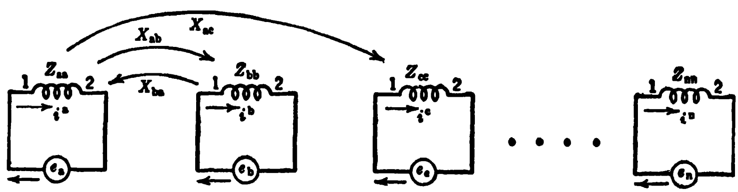

FIG. 2·12. Primitive Mesh Network.

The network, whose equations are the simplest to establish and which serves as a standard reference network, is called the **primitive network.** A given network, whose equation of performance is required, will be called the **derived network.** The derived network will consist of n coils and k meshes where, in general, $n \neq k$. Figures 2·13*a* and 2·14*a* represent derived networks.

The primitive network consists of the following objects and relations: (*a*) n physically separate coils, each short-circuited upon itself. Figures 2·13*b* and 2·14*b* represent primitive networks. (*b*) In series with each coil is an applied voltage. Some of these voltages may be zero. (*c*) Each of the n coils is labeled (an ordered set) so that it may be identified in the new network. (*d*) The ends of each coil of the primitive network are numbered 1 or 2. The positive direction of current and the positive direction of voltage in the primitive network is taken to be from 1 to 2. (*e*) If mutual impedances exist between $k \neq n$ of the n coils of the new network this fact is indicated by arrows on the diagram of the primitive network. In the general case $Z_{ij} \neq Z_{ji}$.

By virtue of the first generalization postulate §2·27 the definition of the primitive network is a natural one. The simplest unit of a k mesh

network is one isolated mesh with impedance and impressed voltage. The equation of performance of this simplest unit is $Z(p)i = e$. In view of the first generalization postulate the equation of performance of the primitive network is $\mathbf{Z} \cdot \mathbf{i} = \mathbf{e}$ or $Z_{mn}i^n = e_m$, where **i**, **e**, and **Z** are properly chosen matrices.

The properly chosen geometric objects of the primitive network are as follows. The n real currents $i^a, i^b, \cdots, i^n$ in the coils of the primitive network will be considered the real components of a fictitious[16] current vector **i**. Its mathematical expression is

$$\mathbf{i} = \begin{array}{c} \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \end{array} \\ \boxed{\begin{array}{c|c|c|c} i^a & i^b & \cdots & i^n \end{array}} \end{array} . \qquad [60]$$

The n impressed voltages $e_a, e_b, \cdots, e_n$ in series with the coils represented in Fig. 2·12 will be considered the real components of the fictitious vector **e**. Its mathematical expression is

$$\mathbf{e} = \begin{array}{c} \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \end{array} \\ \boxed{\begin{array}{c|c|c|c} e_a & e_b & \cdots & e_n \end{array}} \end{array} \qquad [61]$$

The geometric object

$$\mathbf{Z} = \begin{array}{c|cccc|} & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{n} \\ \hline \mathbf{a} & Z_{aa} & Z_{ab} & \cdots & Z_{an} \\ \mathbf{b} & Z_{ba} & Z_{bb} & \cdots & Z_{bn} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{n} & Z_{na} & Z_{nb} & \cdots & Z_{nn} \\ \hline \end{array} \qquad [62]$$

The equation of performance is

$$\mathbf{Z} \cdot \mathbf{i} = \mathbf{e} \quad \text{or} \quad Z_{mn}\, i^n = e_m. \qquad [63]$$

2·37. Derivation of the Differential Equations of All-mesh and Mesh Networks. In §2·41 it will be shown that **e**, **i**, **Z**; **e′**, **i′**, **Z′**, the geometric objects of the primitive and new networks respectively, are tensors. However, in this section we shall assume the tensor character

[16] More advanced concepts of tensor analysis show that the vector **i** is not fictitious but it represents the instantaneous stored magnetic energy in the whole system.

of the above objects and explain the procedure for the setup of the differential equations of performance of the systems. Moreover, since the mere rules for obtaining the equations of performance of all-mesh and mesh networks differ in only a few details we shall reduce the procedure to one set of rules applicable to both types of networks.

The setting up of the differential equations of the derived network consists of three steps: (*a*) Establish correctly labeled diagrams of the derived network and its primitive. This is a purely descriptive step. (*b*) Obtain the transformation matrix **C** showing the difference between the two networks. This is the only analytical step involved. The step employs Kirchhoff's laws. (*c*) Establish the equations of the derived network from that of the primitive network with the aid of **C**. This step involves only routine calculations.

Step (*a*). Before the analysis may begin it is necessary to draw a correctly labeled diagram of the given network. This step is necessary in any method of analysis. The rules for this step are as follows:

(1) Draw a diagram of the derived network and label the separate coils $Z_{aa}, Z_{bb}, \cdots, Z_{nn}$. (See Figs. 2·13*a*–2·14*a*.)

(2) Examine the derived network for mutual impedances. Number the ends of a coil, selected at random, in the derived network with the numerals 1-2. If the next coil, in tracing out a closed circuit in the derived network, is connected series aiding (see §2·34), number its ends 1-2. If it is connected series opposing, label it 2-1. If no mutual inductance exists between the two coils they may be labeled arbitrarily 1-2 or 2-1. Label the ends of all coils.

(3) Indicate one impressed voltage in series with each of the n coils. Some of these voltages $e_a, e_b, \cdots, e_n$ may be zero. Indicate by means of arrows the direction in which these impressed voltages (battery, generator, etc.) act. If these arrows are from 1 to 2 of the respective coils, the numerical voltage is positive, otherwise negative. (See examples §2·39.)

(4) Count the junctions, coils, and sub-networks and apply Eq. (59). By means of Eq. (59) the least number of meshes in the networks of Figs. 2·13*a*–2·14*a* are respectively 5 and 3.

(5) *New variables.* Introduce as many new variables $i^{a'}, i^{b'}, \cdots, i^{n'}$ as there are least number of meshes by drawing as many arrows. These arrows may be drawn along a coil or along an impedanceless branch. The direction of each arrow is arbitrary. Label the arrows $i^{a'}$, $i^{b'}$, $\cdots$, $i^{n'}$. The only restriction on the assignment of $i^{a'}, i^{b'}, \cdots, i^{n'}$ is that they be independent, i.e., they must be sufficient to determine all the currents flowing in the remaining branches. In a mesh network $n' = k \neq n$.

(6) *Old variables.* It is helpful (though not necessary) to draw a diagram of the primitive mesh network having n coils $Z_{aa}, Z_{bb}, \cdots, Z_{nn}$ with mutual impedances between some of them, n currents $i^a, i^b, \cdots, i^n$ and n impressed voltages $e_a, e_b, \cdots, e_n$. The positive direction of the voltages is always from 1 to 2.

When in the derived network no impedance appears in series with an impressed voltage, the primitive network assumes an impedance Z with zero value in series with it. Similarly when in the derived network no voltage appears in series with an impedance, the primitive network assumes a voltage e with zero value in series with it.

Step (*b*). To establish the connection tensor **C** the steps are as follows:

(1) On the diagram of the derived network write along each coil the new current that flows in it. To do this apply Kirchhoff's first law, which states that the sum of all currents entering a junction is zero. (See Eqs. 64*a*, §2·39.)

(2) *Relations between the old and new variables.* On the two diagrams there now exist two expressions in terms of different variables for the current through each coil. Equate the two expressions (that is, the two currents flowing in each coil) giving n equations in k unknowns, $i = f(i')$. It must be remembered that the positive old currents $i^a, i^b, \cdots, i^n$ flow from 1 to 2. For the all-mesh network of Fig. 2·13*a* examine Eqs. (64*a*–64*b*), §2·39. For the mesh network of Fig. 2·14*a* examine Eqs. (64*h*), §2·40.

(3) *The* **C** *matrix.* The matrix composed of the coefficients of the new variables is called the connection matrix or the **C** matrix or **C**-matrix. The **C** matrix for the networks of Figs. 2·13*a*–2·14*a* are given by Eqs. (64*c*) and (64*i*), §§2·39 and 2·40 respectively.

With the establishment of the **C** matrix the set of equations $i^m = f(i^{m'})$ may be written as

$$\mathbf{i} = \mathbf{C}\cdot\mathbf{i}' \quad \text{or} \quad i^m = C^m_{m'}\, i^{m'}$$

representing the relations between the old components **i** and the new components **i**′ of the current vector. These relations for the networks of Figs. 2·13*a*–2·14*a* are given respectively by Eqs. (64*b*) and (64*h*).

Step (*c*). To establish the equations of the given system the steps are as follows:

(1) *Geometric objects of the primitive network.* The three geometric objects **e**, **Z**, and **i** of the general primitive mesh network are given in §2·36. For illustrative examples represented in Figs. 2·13*a*–2·14*a* the geometric objects of the primitive network are given by Eqs. (64*d*)

and (64j), §§2·39 and 2·40 respectively. The equations of the primitive network are

$$\mathbf{Z}\cdot\mathbf{i} = \mathbf{e} \quad \text{or} \quad Z_{mn}\, i^n = e_m.$$

(2) The impedance tensor $\mathbf{Z}'$ or $Z_{m'n'}$ of the derived network is found by the formula

$$\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} \quad \text{or} \quad Z_{m'n'} = Z_{mn}C^m_{m'}C^n_{n'}.$$

These relations for the networks of Figs. 2·13a–2·14a are given by Eqs. (64e) and (64k).

(3) The impressed voltage vector $\mathbf{e}'$ or $e_{m'}$ of the derived network is

$$\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e} \quad \text{or} \quad e_{m'} = C^m_{m'}\, e_m.$$

These relations for Figs. 2·13a–2·14a are Eqs. (64f) and (64l).

(4) The equation of voltage or equation of performance or differential equations of the derived system are

$$\mathbf{Z}'\cdot\mathbf{i}' = \mathbf{e}' \quad \text{or} \quad Z_{m'n'}\, i^{n'} = e_{m'}.$$

This set of differential equations may be subjected to various manipulations depending on the problem at hand.

2·38. Solution of Equations of Performance. If the components of $\mathbf{e}'$ are known, the unknown currents are found by $\mathbf{i}' = (\mathbf{Z}')^{-1}\cdot\mathbf{e}'$. Once the components of $\mathbf{i}'$ have been found then: (a) The currents in each coil are found by $\mathbf{i}_c = \mathbf{C}\cdot\mathbf{i}'$. ($b$) The differences of potential appearing across each coil are $\mathbf{e}_c = \mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'$, where $\mathbf{Z}\cdot\mathbf{C}$ has already been calculated as a step in finding $\mathbf{Z}'$.

2·39. Illustrative Example: All-mesh Network. Obtain the equation of performance of the network represented and described in Figs. 2·13. The explanation of the solution is given in the rules of procedure in §2·37. The mutual inductances are: Z_{aa} and Z_{bb} series aiding, Z_{bb} and Z_{cc} series opposing, Z_{aa} and Z_{dd} series aiding. The absolute values of the impressed voltages e_a, e_b, e_c, e_d, e_f are 3, 4, 7, sin t, cos t. Their directions are indicated on Fig. 2·13a. $e_a = -3$, $e_b = -4$, $e_c = 7$, $e_d = -\sin t$, $e_f = -\cos t$.

In three of the coils of Fig. 2·13a new variables have been assumed. Kirchhoff's first law applied to Fig. 2·13a yields Fig. 2·13c. From Fig. 2·13c the current in coil

$$\begin{aligned} Z_{aa} &\text{ is } i^{a'} - i^{b'} - i^{f'}, \\ Z_{ff} &\text{ is } i^{f'} - i^{c'} - i^{d'}, \\ Z_{bb} &\text{ is } i^{b'} + i^{d'} + i^{c'}. \end{aligned} \qquad [64a]$$

Remembering that i^a, i^b, i^c, i^d, and i^f flow from 1 to 2 and equating the currents flowing in each coil (compare Figs. 2·13*b* and 2·13*c* and use Eqs. 64*a*), we find that the current in coil

$$
\begin{aligned}
Z_{aa} &\text{ is } i^a = i^{a'} - i^{b'} + 0 + 0 - i^{f'}, \\
Z_{bb} &\text{ is } i^b = 0 - i^{b'} - i^{c'} - i^{d'} + 0, \\
Z_{cc} &\text{ is } i^c = 0 + 0 + i^{c'} + 0 + 0, \\
Z_{dd} &\text{ is } i^d = 0 + 0 + 0 - i^{d'} + 0, \\
Z_{ff} &\text{ is } i^f = 0 + 0 - i^{c'} - i^{d'} + i^{f'}.
\end{aligned}
\qquad [64b]
$$

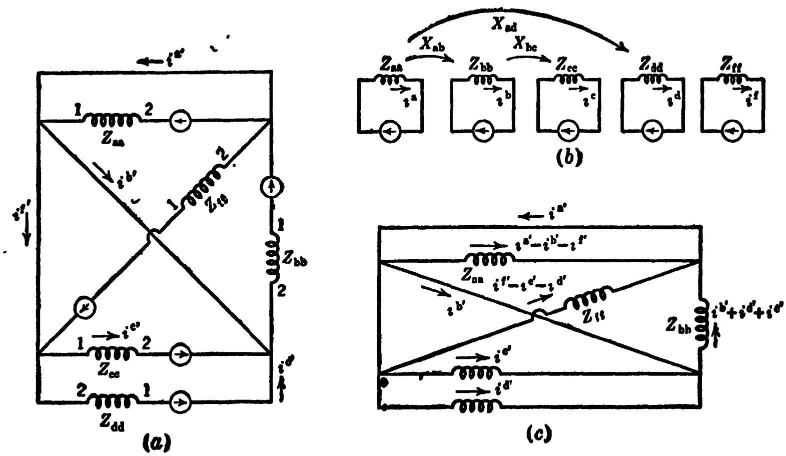

FIG. 2·13. All-mesh Network.

The C-transformation tensor is found by taking the coefficients of the new currents. It is

$$
\mathbf{C} = \begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{f} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathbf{a'} & \mathbf{b'} & \mathbf{c'} & \mathbf{d'} & \mathbf{f'} \end{array} \\
\left|\begin{array}{ccccc}
1 & -1 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 1
\end{array}\right|
\end{array}
\qquad [64c]
$$

The current, voltage, and impedance tensors of the primitive network are

$$
\mathbf{i} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline i^a & i^b & i^c & i^d & i^f \\ \hline \end{array} \end{array}
$$

$$
\mathbf{e} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline e_a & e_b & e_c & e_d & e_f \\ \hline \end{array} \end{array} = \begin{array}{c} \begin{array}{ccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline -3 & -4 & 7 & -\sin t & -\cos t \\ \hline \end{array} \end{array}
$$

$$
\mathbf{Z} = \begin{array}{c|c|c|c|c|c|} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{f} \\ \hline \mathbf{a} & Z_{aa} & X_{ab} & 0 & X_{ad} & 0 \\ \hline \mathbf{b} & X_{ba} & Z_{bb} & 0 & 0 & 0 \\ \hline \mathbf{c} & 0 & 0 & Z_{cc} & & 0 \\ \hline \mathbf{d} & X_{da} & 0 & & Z_{dd} & 0 \\ \hline \mathbf{f} & 0 & 0 & 0 & 0 & Z_{ff} \\ \hline \end{array} \qquad [64d]
$$

The impedance tensor $\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} = Z_{mn}\, C^m_{m'}\, C^n_{m'}$ of the new network is

$$
\mathbf{Z}' = \begin{array}{c|c|c|c|c|c|} & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' \\ \hline \mathbf{a}' & Z_{aa} & -Z_{aa} - X_{ab} & -X_{ab} & -X_{ab} - X_{ad} & -Z_{aa} \\ \hline \mathbf{b}' & -Z_{aa} - X_{ab} & Z_{aa} + X_{ab} + X_{ab} + Z_{bb} & X_{ab} + Z_{bb} & X_{ab} + Z_{bb} + X_{ad} & Z_{aa} + X_{ab} \\ \hline \mathbf{c}' & -X_{ab} & X_{ab} + Z_{bb} & Z_{bb} + Z_{cc} + Z_{ff} & Z_{bb} + Z_{ff} & X_{ab} - Z_{ff} \\ \hline \mathbf{d}' & -X_{ab} - X_{ad} & X_{ab} + X_{ad} + Z_{bb} & Z_{bb} + Z_{ff} & Z_{bb} + Z_{dd} + Z_{ff} & X_{ab} + X_{ad} - Z_{ff} \\ \hline \mathbf{f}' & -Z_{aa} & Z_{aa} + X_{ab} & X_{ab} - Z_{ff} & X_{ab} + X_{ad} - Z_{ff} & Z_{aa} + Z_{ff} \\ \hline \end{array} \qquad [64e]
$$

The voltage vector of the new network is

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & 0 & 0 & 0 \\ \hline -1 & -1 & 0 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & -1 \\ \hline -1 & 0 & 0 & 0 & 1 \\ \hline \end{array} \cdot \begin{array}{|c|}\hline e_a \\ \hline e_b \\ \hline e_c \\ \hline e_d \\ \hline e_f \\ \hline \end{array} = \begin{array}{c|l|}\hline \mathbf{a} & e_a \\ \hline \mathbf{b} & -e_a - e_b \\ \hline \mathbf{c} & -e_b + e_c - e_f \\ \hline \mathbf{d} & -e_b - e_d - e_f \\ \hline \mathbf{f} & -e_a + e_f \\ \hline \end{array}$$

$$= \begin{array}{c|l|}\hline \mathbf{a}' & -3 \\ \hline \mathbf{b}' & 3 + 4 \\ \hline \mathbf{c}' & 4 + 7 - \cos t \\ \hline \mathbf{d}' & 4 + \sin t - \cos t \\ \hline \mathbf{f}' & 3 + \cos t \\ \hline \end{array} \qquad [64f]$$

The differential equations of performance are

$$\mathbf{Z}' \cdot \mathbf{i}' = \mathbf{e}' \quad \text{or} \quad Z_{m'n'}\, i^{n'} = e_{m'}. \qquad [64g]$$

(*c*)

MESH NETWORKS

2·40. Constraints. In § 2·35 it was mentioned that most practical networks contain more coils than meshes. A network possessing more coils than meshes is called a **mesh network.** Mesh networks can be viewed as special cases of all-mesh networks. A mesh network can be considered as an all-mesh network with certain meshes open-circuited (frictionless constraints). The theory is introduced by means of an example. Obtain the equation of performance of the mesh network represented and described in Fig. 2·14. The explanation of the solution is given in the rules of procedure in § 2·37. The coils Z_{mm} and Z_{nn} are wound such that i^m and i^n produce additive values of the

flux. The absolute values of the impressed voltages e_m, e_n, e_p, e_q are 7, 1, 2, 4. Their directions are indicated on the figure.

By Kirchhoff's first law and Fig. 2·14a we have

$$\begin{aligned} i^m &= i^{m'} + 0 \quad + 0, \\ i^n &= 0 \quad + i^{p'} + i^{q'}, \\ i^p &= 0 \quad + i^{p'} + 0, \\ i^q &= 0 \quad + 0 \quad - i^{q'}, \end{aligned} \tag{64h}$$

where it is remembered that i^m, i^n, i^p, i^q flow from 1 to 2.

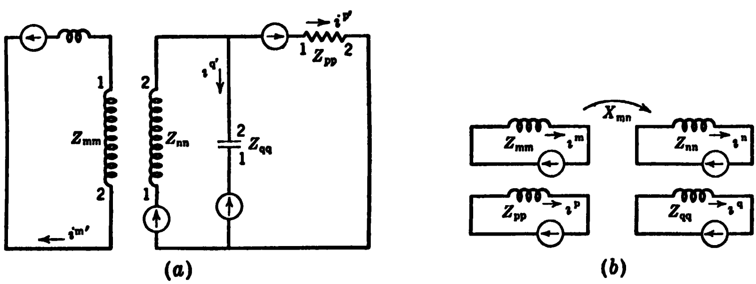

FIG. 2·14. Mesh Network with Sub-network.

The **C** tensor is

$$\mathbf{C} = C^{\alpha}_{\alpha'} = \begin{array}{c|ccc|} & \mathbf{m'} & \mathbf{p'} & \mathbf{q'} \\ \hline \mathbf{m} & 1 & 0 & 0 \\ \mathbf{n} & 0 & 1 & 1 \\ \mathbf{p} & 0 & 1 & 0 \\ \mathbf{q} & 0 & 0 & -1 \\ \hline \end{array} \tag{64i}$$

The current, voltage, and impedance tensors of the primitive network are respectively

$$\mathbf{i} = \begin{array}{|c|c|c|c|} \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} \\ \hline i^m & i^n & i^p & i^q \\ \hline \end{array} \qquad \mathbf{e} = \begin{array}{|c|c|c|c|} \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} \\ \hline e_m & e_n & e_p & e_q \\ \hline \end{array}$$

$$\mathbf{Z} = Z_{mn} = \begin{array}{c|c|c|c|c|} & \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} \\ \hline \mathbf{m} & Z_{mm} & X_{mn} & 0 & 0 \\ \hline \mathbf{n} & X_{nm} & Z_{nn} & 0 & 0 \\ \hline \mathbf{p} & 0 & 0 & Z_{pp} & 0 \\ \hline \mathbf{q} & 0 & 0 & 0 & Z_{qq} \\ \hline \end{array} \qquad [64j]$$

The impedance tensor $Z' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$ or $Z_{\alpha'\beta'} = Z'_{mn}\, C^m_{\alpha'}\, C^n_{\beta'}$ of the new network is

$$\mathbf{Z}' = Z_{\alpha'\beta'} = \begin{array}{c|c|c|c|} & \mathbf{m}' & \mathbf{p}' & \mathbf{q}' \\ \hline \mathbf{m}' & Z_{mm} & X_{mn} & X_{mn} \\ \hline \mathbf{p}' & X_{nm} & Z_{nn} + Z_{pp} & Z_{nn} \\ \hline \mathbf{q}' & X_{mn} & Z_{nn} & Z_{nn} + Z_{qq} \\ \hline \end{array} \qquad [64k]$$

The voltage vector $\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e}$ or $e_{\alpha'} = C^{\alpha}_{\alpha'} e_\alpha$ of the new network is

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} = e_{\alpha'} = \begin{array}{|c|} \hline e_m \\ \hline e_n + e_p \\ \hline e_n - e_q \\ \hline \end{array} = \begin{array}{|c|} \hline -7 \\ \hline 3 \\ \hline -3 \\ \hline \end{array} \qquad [64l]$$

The differential equation of performance or the equation of performance is

$$\mathbf{Z}' \cdot \mathbf{i}' = \mathbf{e}' \quad \text{or} \quad Z_{\alpha'\beta'} i^{\beta'} = e_{\alpha'}. \qquad [64m]$$

EXERCISES X

1. Obtain the differential equations of performance of the network shown in Fig. 2·15.

The mutual inductances are:

Z_{aa} and Z_{bb} series aiding,
Z_{bb} and Z_{cc} series opposing,
Z_{aa} and Z_{dd} series aiding.

The absolute values of the impressed voltages are $e_a = 1$, $e_b = f(t)$, $e_c = E \sin t$, $e_d = 0$, $e_f = -4$. Their directions are indicated on the figure.

2. In the example of §2·40 let the values of the coil impedances be

$$Z_{mn} = L_{nm}p = p, \quad Z_{pp} = R_{pp} = 0.2, \quad X_{mn} = M_{mn}\,p = 0.5p,$$

$$Z_{nn} = L_{nn}p = 2p, \quad Z_{qq} = \frac{1}{C_{qq}p} = \frac{1}{0.008p}.$$

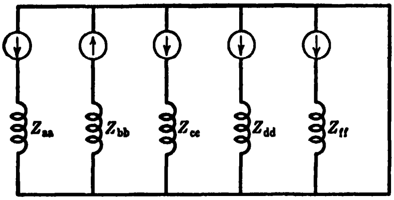

FIG. 2·15. All-mesh Network.

Obtain the matrix solution for $\mathbf{i}'$ of $\mathbf{Z}'\cdot\mathbf{i}' = \mathbf{e}'$, i.e., Eqs. (64m).

3. The voltages induced in the individual coils of the network are given by $\mathbf{Z}\cdot\mathbf{i} = \mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'$. This matrix giving these voltages is denoted by $\mathbf{e}$. Compute $\mathbf{e}$ for the network of §2·40.

4. In the illustrative example of §2·39 let the values of the coil impedances be

$$\begin{aligned} Z_{aa} &= L_{aa}\,p = p, & Z_{ff} &= L_{ff}\;p = 0.5p, \\ Z_{bb} &= L_{bb}\,p = 2p, & X_{ab} &= M_{ab}\,p = 0.2p, \\ Z_{cc} &= L_{cc}\,p = 3p, & X_{bc} &= M_{bc}\,p = 0.3p, \\ Z_{dd} &= L_{dd}\,p = 0.4p, & X_{ad} &= M_{ad}\,p = 0.1p. \end{aligned}$$

Obtain the values of the voltages induced in each coil of the network.

5. Set up the differential equations of performance for the network of Fig. 2·16.

6. Obtain the differential equations of performance of the network of Fig. 2·17.

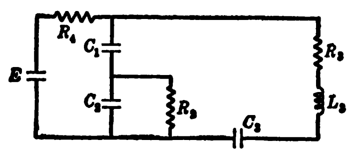

FIG. 2·16. Mesh Network.

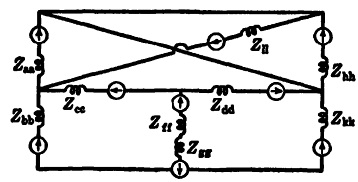

FIG. 2·17. Mesh Network.

The mutual impedance relations are as follows: The pairs of coils $Z_{aa} - Z_{bb}$, $Z_{cc} - Z_{dd}$, $Z_{dd} - Z_{hh}$, $Z_{hh} - Z_{ll}$, $Z_{kk} - Z_{gg}$ are connected series opposing. The coils $Z_{ff} - Z_{gg}$ are connected series aiding. The absolute value of e_a is unity. All other voltages are zero.

7. If no mutual impedance exists between any of the coils of the network of Ex. 6, write the differential equations of performance. The direction of the impressed voltages are shown in the figure. The absolute values of the voltages are $e_a = 1$. The remaining voltages are zero.

2·41. Transformation Formulas for i, e, Z. The geometric objects **i**, **e**, **Z** are now shown to be tensors. In §2·35 it has been pointed out that the instantaneous power input P of the primitive network has the same value as the instantaneous power input P' of the new all-

mesh network. This fact gives the relation $\mathbf{e}\cdot\mathbf{i} = \mathbf{e}'\cdot\mathbf{i}'$ or $e_m i^m = e_{m'} i^{m'}$.

To prove that **i**, **e** and **Z** are tensors we have the following relations:

1. $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ or $i^m = C^m_{m'} i^{m'}$, [65]

2. $\mathbf{e}\cdot\mathbf{i} = \mathbf{e}'\cdot\mathbf{i}'$ or $e_m i^m = e_{m'} i^{m'}$, [66]

3. Second generalization postulate.

We now readily obtain:

(*a*) *Current transformation formula.* Equation (65) is the transformation formula for the current. For an all-mesh network **C** is non-singular and we have

$$\mathbf{i} = \mathbf{C}\cdot\mathbf{i}' \quad \text{or} \quad i^m = C^m_{m'} i^{m'}. \tag{67}$$

(*b*) *Voltage transformation formula.* Substituting the value of **i** from Eq. (65) in Eq. (66) we have

$$\mathbf{e}\cdot\mathbf{C}\cdot\mathbf{i}' = \mathbf{e}'\cdot\mathbf{i}' \quad \text{or} \quad e_m C^m_{m'} i^{m'} = e_{m'} i^{m'},$$

or

$$\mathbf{e}\cdot\mathbf{C} = \mathbf{e}' \quad \text{or} \quad e_m C^m_{m'} = e_{m'}. \tag{68}$$

(*c*) *Transformation formula of Z_{mn}.* The equation of performance of the primitive network is

$$\mathbf{e} = \mathbf{Z}\cdot\mathbf{i}'.$$

Substitution of the values of **i** and **e** from Eqs. (67) and (68) in the above equation yields

$$\mathbf{C}_t^{-1}\cdot\mathbf{e}' = \mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}.$$

Multiplying the equation by $\mathbf{C}_t$ we have

$$\mathbf{e}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C}\cdot\mathbf{i}'.$$

By the second generalization postulate $\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}'$. From this equation and the expression for $\mathbf{e}'$ above it follows that

$$\mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} \quad \text{or} \quad Z_{m'n'} = C^m_{m'} C^n_{n'} Z_{mn}. \tag{69}$$

This is the transformation formula for Z_{mn}.

(*d*)

INTERCONNECTION OF NETWORKS

The methods, thus far developed, are of outstanding value in the interconnection of networks. If a large network is composed of a finite number of simple or complicated networks each of the smaller networks may be analyzed separately and use of these analyses made after the

smaller networks are interconnected into the super-network. This is especially advantageous if the network can be divided up functionally, i.e., in such a manner that all circuits performing similar functions can be grouped together.

It is emphasized that if each of the smaller networks have been analyzed it is not necessary, by the present methods, to start *ab initio*

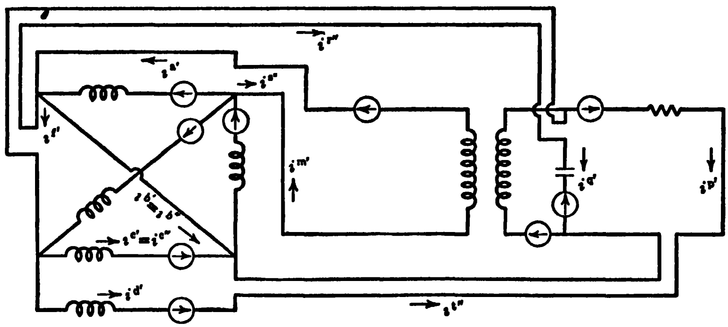

FIG. 2·17*a*. Interconnected Networks.

in the analysis of the super-network. All analyses of the smaller networks can be employed without change.

The procedure is evident from the solution of an example.

2·42. Description of Illustrative Example. It is required to set up the equation of performance of the network of Fig. 2·17*a*. The component parts of this network are the two networks shown in Figs. 2·13*a*–

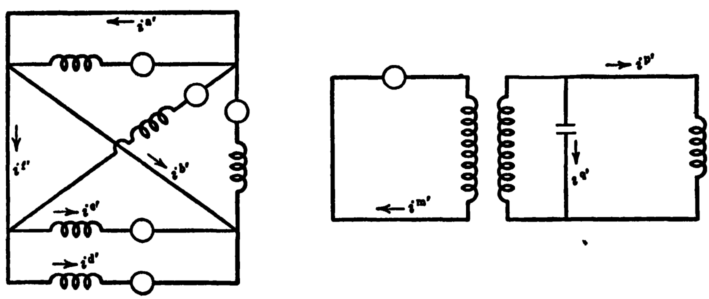

FIG. 2·17*b*. Primitive Network for Interconnected Networks.

2·14*a*. The interconnections are as shown in the figure. The mutual impedances in Fig. 2·17*a* are the same as in the networks of Figs. 2·13*a*–2·14*a*. The same statement holds regarding the voltages.

The primitive network of Fig. 2·17*a* is shown in Fig. 2·17*b* consisting of the two original networks (Figs. 2·13*a*–2·14*a*) placed side by side.

2·43. Geometric Objects of Primitive Network. The impedance tensor of the network of Fig. 2·13*a* is given by Eq. (64*e*). The impedance tensor of the network of Fig. 2·14*a* is given by Eq. (64*k*). Denote these two tensors by $\mathbf{Z}'_1$ and $\mathbf{Z}'_2$ respectively. The impedance tensor of the primitive network (Fig. 2·17*b*) is the sum of these two tensors.

$$\mathbf{Z}' = \mathbf{Z}'_1 + \mathbf{Z}'_2 = \begin{array}{c|cccccccc|}
 & \mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' & \mathbf{m}' & \mathbf{p}' & \mathbf{q}' \\
\hline
\mathbf{a}' & Z_{a'a'} & X_{a'b'} & X_{a'c'} & X_{a'd'} & X_{a'f'} & & & \\
\mathbf{b}' & X_{b'a'} & Z_{b'b'} & X_{b'c'} & X_{b'd'} & X_{b'f'} & & & \\
\mathbf{c}' & X_{c'a'} & X_{c'b'} & Z_{c'c'} & X_{c'd'} & Z_{c'f'} & & & \\
\mathbf{d}' & X_{d'a'} & X_{d'b'} & X_{d'c'} & Z_{d'd'} & X_{d'f'} & & & \\
\mathbf{f}' & X_{f'a'} & X_{f'b'} & X_{f'c'} & X_{f'd'} & Z_{f'f'} & & & \\
\mathbf{m}' & & & & & & Z_{m'm'} & X_{m'p'} & X_{m'q'} \\
\mathbf{p}' & & & & & & X_{p'm'} & Z_{p'p'} & X_{p'q'} \\
\mathbf{q}' & & & & & & X_{q'm'} & X_{q'p'} & Z_{q'q'} \\
\hline
\end{array} \qquad [70]$$

The value of the components of $\mathbf{Z}'$ in terms of the components of the individual networks is found by comparison of Eq. (70) with Eqs. (64*e*–64*k*). For example

$$Z_{a'b'} = - Z_{aa} - X_{ab} \quad \text{and} \quad Z_{p'p'} = Z_{nn} + Z_{pp}.$$

The voltage vectors of networks (2·13*a*–2·14*a*) are given by Eqs. (64*f*–64*l*). Denote these by $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. The voltage vector of their primitive is their sum,

$$\mathbf{e}' = \mathbf{e}'_1 + \mathbf{e}'_2 = \begin{array}{|cccccccc|}
\mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' & \mathbf{m}' & \mathbf{p}' & \mathbf{q}' \\
\hline
e_{a'} & e_{b'} & e_{c'} & e_{d'} & e_{f'} & e_{m'} & e_{p'} & e_{q'} \\
\hline
\end{array} \qquad [71]$$

where, for example, e'_p is identified by means of Eq. (64*l*) to be $e_n + e_p$.

Likewise the current vector is

$$\mathbf{i}' = \mathbf{i}'_1 + \mathbf{i}'_2 = \begin{array}{|cccccccc|}
\mathbf{a}' & \mathbf{b}' & \mathbf{c}' & \mathbf{d}' & \mathbf{f}' & \mathbf{m}' & \mathbf{p}' & \mathbf{q}' \\
\hline
i^{a'} & i^{b'} & i^{c'} & i^{d'} & i^{f'} & i^{m'} & i^{p'} & i^{q'} \\
\hline
\end{array} \qquad [72]$$

2·44. The Transformation Tensor. We are now ready to interconnect the two networks as indicated in Fig. 2·17*a*. Introduce five new currents $i^{b''}$, $i^{c''}$, $i^{r''}$, $i^{s''}$, $i^{t''}$ shown in the figure as there are only five meshes. From the figure it is clear that

in coils where formerly $i^{f'}$ and $i^{q'}$ flowed, now $i^{r''}$ flows,

in coils where formerly $i^{a'}$ and $i^{m'}$ flowed, now $i^{s''}$ flows, [73]

and

in coils where formerly $i^{d'}$ and $i^{p'}$ flowed, now $i^{t''}$ flows.

It is evident from Eq. (73) that the nine currents $i^{a'}$, $i^{b'}$, $i^{c'}$, $i^{d'}$, $i^{e'}$, $i^{f'}$, $i^{m'}$, $i^{n'}$, $i^{q'}$ can be expressed in terms of five new currents. Relations (73) and the figure yield the relations of Eq. (74) from which the **C** transformation tensor is found

$$\begin{array}{lccccc}
 & \mathbf{b''} & \mathbf{c''} & \mathbf{r''} & \mathbf{s''} & \mathbf{t''} \\
i^{a'} = & 0 & +\,0 & +\,0 & +\,i^{s''} & +\,0 \\
i^{b'} = & i^{b''} & +\,0 & +\,0 & +\,0 & +\,0 \\
i^{c'} = & 0 & +\,i^{c''} & +\,0 & +\,0 & +\,0 \\
i^{d'} = & 0 & +\,0 & +\,0 & +\,0 & +\,i^{t''} \\
i^{f'} = & 0 & +\,0 & +\,i^{r''} & +\,0 & +\,0 \\
i^{m'} = & 0 & +\,0 & +\,0 & -\,i^{s''} & +\,0 \\
i^{p'} = & 0 & +\,0 & +\,0 & +\,0 & -\,i^{t''} \\
i^{q'} = & 0 & +\,0 & +\,i^{r''} & +\,0 & +\,0
\end{array} \qquad [74]$$

$$\mathbf{C}'' = \begin{array}{c|ccccc|}
 & \mathbf{b''} & \mathbf{c''} & \mathbf{r''} & \mathbf{s''} & \mathbf{t''} \\
\hline
\mathbf{a'} & 0 & 0 & 0 & 1 & 0 \\
\mathbf{b'} & 1 & 0 & 0 & 0 & 0 \\
\mathbf{c'} & 0 & 1 & 0 & 0 & 0 \\
\mathbf{d'} & 0 & 0 & 0 & 0 & 1 \\
\mathbf{f'} & 0 & 0 & 1 & 0 & 0 \\
\mathbf{m'} & 0 & 0 & 0 & -1 & 0 \\
\mathbf{p'} & 0 & 0 & 0 & 0 & -1 \\
\mathbf{q'} & 0 & 0 & 1 & 0 & 0 \\
\hline
\end{array} \qquad [75]$$

2·45. Geometric Objects of New Network. The voltage vector of the new network is

$$e_{\alpha''} = C^{\alpha'}_{\alpha''}\, e_{\alpha'} = \mathbf{C}''_t \cdot \mathbf{e}',$$

where $C^{\alpha'}_{\alpha''}$ and $e_{\alpha'}$ are given respectively by Eqs. (75–71). (See Ex. 1.)

The impedance tensor $Z_{\alpha''\beta''}$ is given by

$$Z_{\alpha''\beta''} = Z_{m'n'}\, C^{m'}_{\alpha''}\, C^{n'}_{\beta''} = \mathbf{C}''_t \cdot \mathbf{Z}' \cdot \mathbf{C}''. \qquad [76]$$

The equation of performance is

$$Z_{\alpha''\beta''}\, i^{\beta''} = e_{\alpha''}. \qquad [76a]$$

The solution of Eq. (76*a*) yields $i^{b''}, i^{c''}, i^{r''}, i^{s''}, i^{t''}$. These values substituted in Eq. (74) give $i^{a'}, i^{b'}, i^{c'}, i^{d'}, i^{f'}, i^{m'}, i^{p'}, i^{q'}$. Finally, these currents when substituted in Eq. (64*b*) and Eq. (64*h*) yield the currents passing through the individual coils of the network.

EXERCISES XI

1. Compute $e_{\alpha''}$ by carrying out the multiplications indicated in $e_{\alpha''} = C_{\alpha''}^{\alpha'} e_{\alpha'}$.
2. Compute $Z_{\alpha''\beta''}$ by carrying out the operations indicated in Eq. (76).
3. Solve Eq. (76*a*) for $i^{s''}$.

PART (B)

INTRODUCTION TO TENSOR ANALYSIS OF ROTATING ELECTRICAL MACHINERY

Sections (1–4) of the present chapter consist of an introduction to certain parts of tensor analysis of general linear networks where the nature of the coils has not been considered. They may have been stationary or rotating coils. Sections (5–10) of this chapter are devoted to a brief introduction to certain portions of tensor analysis of rotating electrical machinery.

(5)

Non-mathematical Outline of the Nature of the Theory of Rotating Electrical Machinery

Physically, a rotating electrical machine is but two electromagnetic-mechanical configurations composed of non-magnetic and magnetic materials and mesh networks (with their coexisting magnetic fields) such that relative rotary motion of various velocities is possible between the configurations. The configurations differ in many details and consequently there are many different types of machines. Examples of types of machines are: shunt direct-current motors, synchronous generators, induction motors, repulsion motors, etc. However, when analyzed from the proper point of view (tensor viewpoint) the many different types of machines (called derived machines in this chapter) are strikingly similar and may be considered mere aspects of one primitive machine.

2·46. Scope. Tensor analysis of rotating electrical machinery is a large field. The published work in this field is very extensive. In Secs. (5–9) the motion of the rotor is assumed to be known and the analysis is for purely electrodynamic systems. The amount of tensor analysis required in these sections is restricted to definitions, tensor transformation formulas, certain properties of transformations, tensor addition, subtraction, inner product. The theory is applied to a number of derived machines.

The theory in Sec. 10 is more condensed than the work in the preceding sections, but many references to the original papers of Kron are given. In this section the motion of the rotor is, in general, unknown. The system (or systems) is an electrodynamic-mechanical one and advanced tensor analysis and advanced geometrical concepts are employed. The most general equation of performance of rotating electrical machinery is developed. This equation is valid for the most general situation possible in the analysis of one or a system of rotating electrical machines. One of the purposes of the development of the general equation of motion is the study of acceleration in all types of rotating electrical machines. Another use of the general equation is the analysis of hunting of machines. Closely associated with the last analysis is the study of stability of machine systems.

2·47. Preliminary Description of Primitive Machines. It has been noted in Sec. 4 that every mesh network consists of n coils interconnected. The connections may be electromagnetic or conductive. No matter how complex the mesh network, it has been made to depend upon the primitive network of §2·36 consisting of n distinct coils possessing no electrical connections between them. They may have magnetic or dielectric connections between them. Any particular network (called a derived network) can be built physically from the n coils of the primitive network and the equation of performance of the derived network can be obtained from the tensor concepts and the **C** connection or transformation tensor. The number of types of mesh networks is very large. A classification of such according to function, characteristics, or application would be a tedious task. Examples are: bridge circuits, two- and multiple-winding transformers, auto-transformers, transmission and filter networks, and armature windings. From a tensor viewpoint all these are but aspects of one primitive network.

It is then anticipated that Kron's tensor analysis of rotating machines will proceed along similar lines. As reasonably expected the analysis is much more complicated than that of stationary mesh networks because there enters the complexities introduced by various relative motions between magnetically coupled circuits. There are two

sets of axes: one pertaining to the stator of the machine, the other belonging to the rotor. At least one primitive machine is expected. In fact because of computational exigencies there are two primitive machines. The distinction between them is one of difference of preferred reference frames and the consequences resulting therefrom.

The first primitive machine is called the **primitive machine with stationary reference axes.** The reference axes on both stator and rotor of this machine are stationary in space, i.e., fixed relative to the base of the machine. For reasons later evident, this machine will be called also the non-holonomic (or rather quasi-holonomic) machine. (See Chap. I, Sec. 6, for non-holonomic dynamical systems and coordinates.) There are associated with this primitive machine, as for the primitive mesh network, certain fundamental tensors called the resistance, inductance, and torque tensors. *The components of these tensors for the quasi-holonomic machine are constants.* This fact reduces many analyses of many rotating machines to the simplicity of linear stationary network analyses. It is possible to base the derivation of the equations of performance of the non-holonomic machine on Maxwell's field equations. It is also possible to establish its equations of performance by means of ingenious physical concepts of Kron. The latter method will be followed here.

The holonomic primitive machine or second primitive machine differs from the non-holonomic machine in the following respects: The reference axes on the stator remain fixed, but the reference axes on the rotor are rotating axes. The speed of rotation of the rotor and axes are identical. The components of the fundamental tensors associated with this machine are, in general, no longer constants but functions of θ, the angular displacement of the rotor. The equations of performance of the holonomic machine can be derived directly from the equations of Lagrange, provided the rotating axes move at the same speed as the rotor.

The equations of performance of either primitive machine are derivable from the equations of performance of the other primitive machine by change of reference systems.

2·48. Derived Machines. Derived rotating electrical machines are classes or types of rotating machines such as: salient-pole synchronous motors or generators, direct-current motors or generators, repulsion motors, squirrel-cage motors, and Schrage motors. Derived machines are analogous to derived stationary linear mesh networks of Sec. 4. The equations of performance of a derived machine are obtained from the equations of performance of one or the other (sometimes both) of the primitive machines by routine manipulations (ma-

trix multiplications) involving the use of a transformation or connection tensor, called the **C** tensor. The **C** tensor is determined by an inspection of and a comparison of the windings of the derived machine with the windings of a primitive machine. There are at least as many **C** tensors as there are derived machines. When the derived machine is obtainable from both the non-holonomic and holonomic machines, then there are two **C** tensors for each derived machine. For each machine also as many additional **C** tensors may be introduced as there are types of artificial or hypothetical reference frames employed. Hypothetical reference frames appear with the use of symmetrical components, magnetizing and load currents, and with other labor-saving concepts.

The old theory or theories of rotating electrical machinery consists of a large number of individual and largely mutually independent theories of each machine based on some original physical picture invented by a specialist in the theory of one derived machine. In general, more than one theory exists for each derived machine, so that the total number of theories closely approximates the total number of specialists. All these piecemeal theories are replaceable by Kron's tensor theory of rotating electrical machinery. Clearly, only a brief introduction to this theory and its application to but few machines can be attained in a single chapter.

(6)

Primitive Machine with Stationary Reference Axes

In this section the general equations of performance of the non-holonomic or quasi-holonomic machine are derived from the physical pictures of Kron. The material is analogous to that of §2·36 on the primitive system of mesh networks.

2·49. The Primitive Machine with Stationary Reference Axes. The first primitive machine has the following characteristics:

(*a*) The stator has two salient-poles. (See Figs. 2·18*a* and 2·18*b*.)

(*b*) The rotor is smooth.

(*c*) All slip-rings, commutators, and electrical connections between any windings of the machine are considered removed. (Compare primitive network §2·36.)

(*d*) The rotor windings are symmetrically distributed around the circumference. There may be any number of them arranged in layers and each winding may have different constants.

(*e*) On the stator, windings exist in the axes of the salient poles and in axes midway between the salient poles. There may be any number

of windings in each of these axes and each winding may have different constants.

(*f*) Along each winding of the stator and of the rotor are two reference axes. One is called the direct axis, denoted by $\mathbf{d}$; the other the quadrature axis, denoted by $\mathbf{q}$. (A description of direct and quadrature axes as applied to the special case of a synchronous machine is given in §2·50. The $\mathbf{d}$ and $\mathbf{q}$ axes on both rotor and stator are fixed in space, i.e., their origin and directions are fixed relative to the base of the machine.

(*g*) Associated with each winding of the rotor are two unit vectors $\mathbf{d}_i$ and $\mathbf{q}_j$. These unit vectors on the stator windings are denoted by

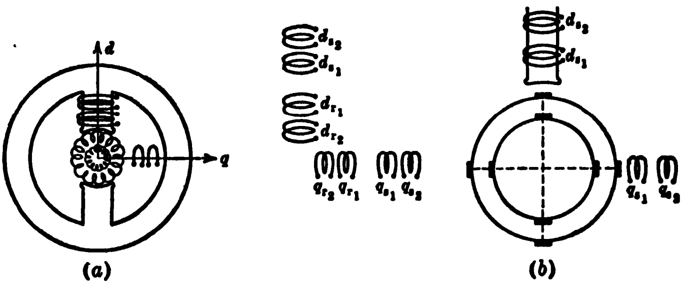

FIG. 2·18. Generalized Rotating Machine.

$\mathbf{d}_{s1}, \mathbf{d}_{s2}, \cdots, \mathbf{d}_{sn}$ and $\mathbf{q}_{s1}, \mathbf{q}_{s2}, \cdots, \mathbf{q}_{sn}$, where $\mathbf{d}_{s1}$ and $\mathbf{q}_{s1}$ belong to the stator winding nearest the air gap of the machine. The unit vectors on the rotor windings are denoted by $\mathbf{d}_{r1}, \mathbf{d}_{r2}, \cdots, \mathbf{d}_{rn}$ and $\mathbf{q}_{r1}, \mathbf{q}_{r2}, \cdots, \mathbf{q}_{rn}$, where $\mathbf{d}_{r1}$ and $\mathbf{q}_{r1}$ belong to the rotor winding nearest the stator. (See Fig. 2·18*b*.)

(*h*) In the theory saturation and iron losses are neglected. It is assumed that the inductance between a stator and rotor coil is a sinusoidal function of the position of the rotor. *However, all formulas developed are valid for any number of harmonics provided only that the tensors and geometric objects are enlarged by the addition of proper components, i.e., by the addition of rows and columns.*

2·50. Two-reaction Coordinates. (A Digression from the Tensor Theory to Direct and Quadrature Quantities Relative to Synchronous Machine Analysis.) The reason for the choice of direct and quadrature coordinates is evident from their application to a salient-pole synchronous machine. Various m.m.f.'s of armature and exciting windings of a machine can be combined vectorially only in case they act upon the same magnetic circuits. It is obvious (Fig. 2·19*a*) that the magnetic circuit formed by armature phase *A* and by the rotor when it is in posi-

tion $\theta = 0$ is much different from that formed by the same circuit and by the rotor when it is in position $\theta = \pi/2$. For vectorial addition (and for other reasons) it is advantageous to choose a reference axis pointing from the center of the rotor along the central line of a field pole. This is called the direct axis *d*. At right angles (90 electrical degrees) ahead of the direct axis is the quadrature axis *q*. In the holonomic approach to synchronous machine theory these axes are moving axes, they being attached or fixed to the field or moving rotor. Of course, due to symmetry existing in all machines it is sufficient to consider the machine to be a two-pole machine. The armature windings of

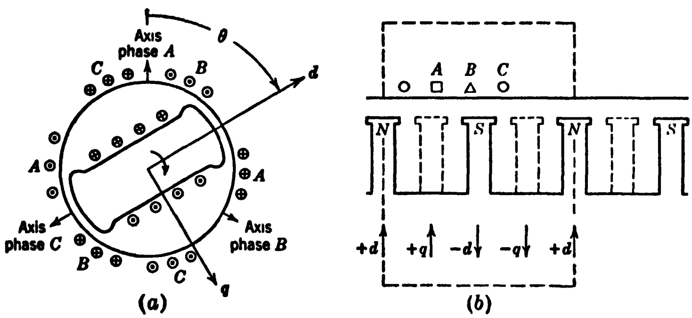

FIG. 2·181. Direct and Quadrature Quantities.

a three-phase winding are represented in Fig. 2·181*a*. Denote the three-phase currents and the three-phase voltages of the machine by i_a, i_b, i_c and e_a, e_b, e_c respectively. Denote the magnetic linkages (self- and mutual) in phases *A*, *B*, and *C* by ψ_a, ψ_b, and ψ_c. The nine variables i_a, i_b, i_c, e_a, e_b, e_c, ψ_a, ψ_b, ψ_c are not fictitious quantities, but actual physical quantities existing in the machine. However, for the reason already stated and for greater mathematical simplicity, which is a consequence of the previous reason, it is convenient to employ nine new variables

i_d, e_d, ψ_d direct-axis current, voltage, linkage

i_q, e_q, ψ_q quadrature-axis current, voltage, linkage

i_0, e_0, ψ_0 zero-axis current, voltage, linkage.

These nine new variables are defined in terms of the old by the equations:

$$i_d = \tfrac{2}{3}[i_a \cos \varphi + i_b \cos (\varphi - 120°) + i_c \cos (\varphi + 120°)].$$

$$i_q = -\tfrac{2}{3}[i_a \sin \varphi + i_b \sin (\varphi - 120°) + i_c \sin (\varphi + 120°)]. \quad [77]$$

$$i_0 = \tfrac{1}{3}[i_a + i_b + i_c].$$

$$\begin{aligned} e_d &= \tfrac{2}{3}[e_a \cos\varphi + e_b \cos(\varphi - 120°) + e_c \cos(\varphi + 120°)]. \\ e_q &= -\tfrac{2}{3}[e_a \sin\varphi + e_b \sin(\varphi - 120°) + e_c \sin(\varphi + 120°)]. \\ e_0 &= \tfrac{1}{3}[e_a + e_b + e_c]. \end{aligned} \qquad [78]$$

$$\begin{aligned} \psi_d &= \tfrac{2}{3}[\psi_a \cos\varphi + \psi_b \cos(\varphi - 120°) + \psi_c \cos(\varphi + 120°)]. \\ \psi_q &= -\tfrac{2}{3}[\psi_a \sin\varphi + \psi_b \sin(\varphi - 120°) + \psi_c \sin(\varphi + 120°)]. \\ \psi_0 &= \tfrac{1}{3}[\psi_a + \psi_b + \psi_c]. \end{aligned} \qquad [79]$$

The values $\pm 2/3$, $1/3$ in Eqs. (77, 78, 79) need not concern us here.

Equations (77) can be solved for i_a, i_b, i_c in terms of i_d, i_q, i_0. Likewise Eqs. (78) and (79) can be solved respectively for e_a, e_b, e_c and ψ_a, ψ_b, ψ_c.

The performance of the machine can be described by means of a system [17] of differential equations some of whose dependent variables are i_d, i_q, e_d, e_q, e_0, i_0. When these differential equations can be solved for the direct-, quadrature-, and zero-axis quantities, then the actual physical quantities, phase voltages, currents, and linkages, are immediately obtained from the inverses of Eqs. (77, 78, 79).

In general, the direct and quadrature quantities are not actual physical quantities. Yet under certain modes of operation and in machines of certain design they may be physical quantities. For example, if the only winding in the field of a synchronous machine is the main field winding then $I_d = I$, the field current of the machine.

2·51. (Tensor Theory Resumed) Equations of a Moving Winding. Let an instantaneous voltage e be impressed on a closed winding moving with instantaneous velocity $p\theta$ in a magnetic field which is produced by an outside current flowing in a stationary winding. At time t all currents vary and the moving winding is accelerated.

It is desired to obtain a differential equation relating to the physical phenomena present in the moving winding. It is necessary to select a space-time reference frame in order to specify the quantities which are to be measured. In the reference frame chosen the observer (measurer) is electrically stationary relative to the moving coil. If a voltmeter is connected to a moving coil through slip-rings R_1 and R_2 then

[17] R. H. Park, "Definition of an Ideal Synchronous Machine and Formula for the Armature Flux Linkages," *General Electric Review*, **31** (1928); R. H. Park, "Two-Reaction Theory of Synchronous Machines, Part I, Generalized Method of Analysis," *Trans. A.I.E.E.*, **48** (1929); B. R. Prentice, "Fundamental Concepts of Synchronous Machine Reactances," *Trans. A.I.E.E.*, **56** (1937); P. L. Alger, "Calculation of Armature Reactances of Synchronous Machines," *Trans. A.I.E.E.*, **47** (1928).

the observer reading the meter is *electrically stationary* relative to the winding or moving coil.

Consider the difference of potential measured between two points P_1 and P_2 fixed in space and such that P_1 and P_2 are in contact with R_1 and R_2 respectively. At any instant four voltages are measurable between P_1 and P_2. These are the (*a*) impressed voltage e, (*b*) resistance drop Ri, (*c*) voltage $\frac{d\varphi}{dt}$, *induced* in the winding due to change of flux linkages φ, (*d*) voltage $\psi\, p\theta$, *generated* by the moving winding, as if all currents were steady and the winding moving with velocity $p\theta$.

The differential equation expressing the relation between these four voltages is

$$e = Ri + \frac{d\varphi}{dt} + (p\theta)\psi. \qquad [80]$$

Equation (80) is the equation of voltage of a moving winding. If Eq. (80) is multiplied by i, the resulting equation

$$ei = Ri^2 + \frac{d\varphi}{dt}i + (p\theta)\psi\, i \qquad [81]$$

is the equation of power flow, where,

ei = instantaneous power input, $\frac{d\varphi}{dt}i$ = rate of increase of stored magnetic energy,

Ri^2 = power heat loss, $(p\theta)\psi\, i$ = mechanical power output (torque $\times$ velocity).

The torque upon the coil is $i\psi$.

By the first generalization postulate §2·27 it has been shown that the equation of performance, $\mathbf{Z}\cdot\mathbf{i} = \mathbf{e}$, of a network of n meshes can be obtained as a generalization of the equation of performance, $Zi = e$, of a single mesh. In an analogous manner, the method of procedure is to modify Eqs. (80–81) (equations of voltage and power flow) for a moving winding so that these equations become the equation of voltage and equation of power-flow of a rotating machine. The final equations, which will be developed in §§2·52–2·56, for the performance of the first primitive machine are

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \frac{d\mathbf{\Phi}}{dt} + p\theta\,\mathbf{\Psi} \quad \text{(Equation of voltage)} \qquad [82]$$

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}\,p\cdot\mathbf{i} + p\theta\mathbf{G}\cdot\mathbf{i} = (\mathbf{R} + \mathbf{L}p + p\theta\,\mathbf{G})\cdot\mathbf{i}$$
$$\text{(Equation of voltage)} \qquad [83]$$

$$\mathbf{i}\cdot\mathbf{e} = \mathbf{i}\cdot\mathbf{R}\cdot\mathbf{i} + \mathbf{i}\cdot\mathbf{L}\,p\cdot\mathbf{i} + p\theta\,\mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} \quad \text{(Equation of power)} \qquad [84]$$

$$f = \mathbf{i}\cdot\mathbf{\Psi} = \mathbf{i}\cdot\mathbf{G}\cdot\mathbf{i} \quad \text{(Equation of torque),} \qquad [85]$$

or in index notation

$$e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + p\theta\,\Psi_m \qquad [86]$$

$$e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + p\theta\,G_{mn}i^n, \qquad [87]$$

$$e_m i^m = R_{mn}i^m i^n + L_{mn}i^m\frac{di^n}{dt} + p\theta\,G_{mn}i^m i^n, \qquad [88]$$

$$f = \Psi_m i^n = G_{mn}i^m i^n. \qquad [89]$$

The tensors to be developed for the primitive machine are: **e**, **R**, **i**, **Φ**, **Ψ**, **L**, and **G** or in index notation e_m, R_{mn}, i^m, φ_m, ψ_m, L_{mn}, and G_{mn}. The constructions of these tensors are based on Kron's physical concepts and pictures explained in §§2·52–2·56, although they can be derived from purely dynamical considerations.

2·52. Replacement of Rotor (Armature) by Two Sets of Coils at Right Angles. A stationary reference axis on a rotor winding can be replaced by a set of stationary brushes, which may be real or fictitious. The line joining the brushes (Fig. 2·19*c*) is a brush axis. Since the current flows in through one brush and out through the other the current flows in only one direction *AB* on one side of the brush axis and in the opposite direction *CD* on the other side of the same axis (Figs. 2·19*a*, *a'*). Evidently, the flux produced by the **q** axis current extends in the direction of the **q** axis. Thus a set of brushes and consequently a set of reference axes may be considered as a coil (Fig. 2·19*b*). Figure 2·19*c* represents the brush axis. Since each rotor layer has both a **d** and **q** axis the two reference axes are replaced by two sets of brushes and these in turn by two coils at right angles. (Figs. 2·19*a'*, *b'*, *c'*.)

A final diagram of a primitive machine having two layers of windings on both the stator and rotor is represented in Fig. 2·18*b*. A four-layer primitive machine thus consists of four sets of coils. The first two sets belong to the stator and these two sets are arranged at right angles as shown in the figure. As many coils exist in each set as there are separate layers of winding on the stator. The stator coils are stationary in space. The second two sets of coils belong to the rotor and they also are arranged at right angles shown in the figure. As many coils be-

long to each rotor set as there are layers of windings on the rotor. The rotor windings have instantaneous velocity $p\theta$ relative to the stator coils. It may be pointed out that although the physical windings of the rotor have velocity relative to the stator the rotor reference axes remain fixed in space since it is evident from Figs. 2·19*a*, *a′* that the flux of the rotor remains constant in direction. Again it is stated that the brushes may be real or they may be fictitious, depending upon the derived machine. There may be no brushes on the derived machine. The concept of a brush merely furnishes a reference axis.

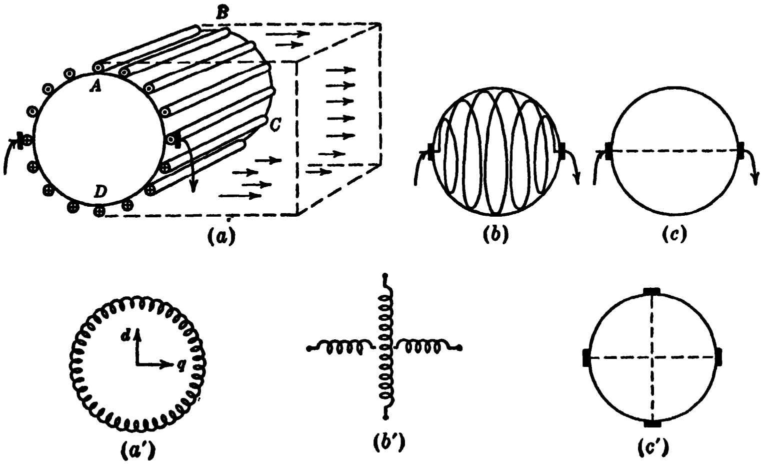

FIG. 2·19. Representations of a Rotor Axis and a Rotor Winding.

2·53. Current, Voltage, Resistance, and Inductance Tensors of the First Primitive Machine. In the theory of stationary networks (Sec. 4) there was only one current and it possessed n components. In the primitive rotating machine there is only one current and it possesses $2m + 2n$ components, where m is the number of layers of winding on the stator and n is the number of layers of windings on the rotor. For simplicity in writing tensors it is assumed in §§2·53–2·58 that $m = n = 1$.

The generalized current is the contravariant vector

$$i^\alpha = \mathbf{i} = i^{ds}\mathbf{d}_s + i^{dr}\mathbf{d}_r + i^{qr}\mathbf{q}_r + i^{qs}\mathbf{q}_s,$$

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \end{array} \\ \mathbf{i} = & \begin{array}{|c|c|c|c|} \hline i^{ds} & i^{dr} & i^{qr} & i^{qs} \\ \hline \end{array} \end{array} \qquad [90]$$

The generalized voltage is the covariant vector

$$e_\alpha = \mathbf{e} = e_{ds}\mathbf{d}_s + e_{dr}\mathbf{d}_r + e_{qr}\mathbf{q}_r + e_{qs}\mathbf{q}_s,$$

$$\mathbf{e} = \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \multicolumn{1}{|c|}{e_{ds}} & \multicolumn{1}{c|}{e_{dr}} & \multicolumn{1}{c|}{e_{qr}} & \multicolumn{1}{c|}{e_{qs}} \\ \hline \end{array} \qquad [91]$$

whose four components are the terminal voltages of the machine, some of which may be zero.

The resistance and inductance tensors representing respectively the resistances and the self- and mutual inductances of the four windings are

$$\mathbf{R} = \begin{array}{c|cccc|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds} & 0 & 0 & 0 \\ \mathbf{d}_r & 0 & r_r & 0 & 0 \\ \mathbf{q}_r & 0 & 0 & r_r & 0 \\ \mathbf{q}_s & 0 & 0 & 0 & r_{qs} \\ \hline \end{array} \quad [92], \qquad \mathbf{L} = \begin{array}{c|cccc|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & L_{ds} & M_d & 0 & 0 \\ \mathbf{d}_r & M_d & L_{dr} & 0 & 0 \\ \mathbf{q}_r & 0 & 0 & L_{qr} & M_q \\ \mathbf{q}_s & 0 & 0 & M_q & L_{qs} \\ \hline \end{array} \quad [93]$$

The resistance drop is $\mathbf{R}\cdot\mathbf{i}$. Each rotor winding is symmetrical and consequently $r_{dr} = r_{qr} = r_r$ for each rotor layer.

2·54. Induced Voltages; Flux Linkage Vector Φ. The linkage vector Φ is given by the equation

$$\Phi = \mathbf{L}\cdot\mathbf{i} = \Phi_{ds}\mathbf{d}_s + \Phi_{dr}\mathbf{d}_r + \Phi_{qr}\mathbf{q}_r + \Phi_{qs}\mathbf{q}_s, \qquad [94]$$

$$= \begin{array}{cccc} \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \multicolumn{1}{|c|}{\phi_{ds}} & \multicolumn{1}{c|}{\phi_{dr}} & \multicolumn{1}{c|}{\phi_{qr}} & \multicolumn{1}{c|}{\phi_{qs}} \\ \hline \end{array}$$

where

$$\begin{aligned} \Phi_{ds} &= i^{ds}L_{ds} + i^{dr}M_d + 0 + 0, \\ \Phi_{dr} &= i^{ds}M_d + i^{dr}L_{dr} + 0 + 0, \\ \Phi_{qr} &= 0 + 0 + i^{qr}L_{qr} + i^{qs}M_q, \\ \Phi_{qs} &= 0 + 0 + i^{qr}M_q + i^{qs}L_{qs}. \end{aligned} \qquad [95]$$

The component Φ_{ds} is the numerical value of the linkages in the $\mathbf{d}_s$ axis due to currents in all axes except those at right angles to $\mathbf{d}_s$. The unit vectors in the column at the left of $\mathbf{L}$ indicate the axis in which the

linkages are taken. Of the four components of $\mathbf{\Phi}$ two are linkages in the stator and two are linkages in the rotor axes. Consequently $\mathbf{\Phi}$ can be written

$$\mathbf{\Phi} = \mathbf{L} \cdot \mathbf{i} = \mathbf{L}_s \cdot \mathbf{i} + \mathbf{L}_r \cdot \mathbf{i}, \tag{96}$$

and

$$\mathbf{\Phi} = \mathbf{L} \cdot \mathbf{i} = (\mathbf{L}_s + \mathbf{L}_r) \cdot \mathbf{i}, \tag{97}$$

where

$$\mathbf{L}_s = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & L_{ds} & M_d & 0 & 0 \\ \mathbf{d}_r & 0 & 0 & 0 & 0 \\ \mathbf{q}_r & 0 & 0 & 0 & 0 \\ \mathbf{q}_s & 0 & 0 & M_q & L_{qs} \end{array} \tag{98}$$

$$\mathbf{L}_r = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & 0 & 0 & 0 & 0 \\ \mathbf{d}_r & M_d & L_{dr} & 0 & 0 \\ \mathbf{q}_r & 0 & 0 & L_{qr} & M_q \\ \mathbf{q}_s & 0 & 0 & 0 & 0 \end{array} \tag{99}$$

The construction of the tensors for the first three terms of each of Eqs. (82–84) or (86–88) are now complete. It remains only to obtain the tensors for the last term of these equations.

2·55. Generated Voltages; Rotor Flux-Density Vector $\mathbf{\Psi}$. The vector $\mathbf{\Psi}$ represents the resultant (due to all currents of the machine) flux density cut by the rotor conductors. In §2·54 the linkage vector $\mathbf{\Phi}$ represents the resultant flux linkages of all windings of the machine.

First it is desired to derive relations between $\mathbf{\Psi}$ and $\mathbf{\Phi}_r$. In this derivation, by means of physical concepts, it is assumed that the flux-density waves are sinusoidally distributed around the circumference in each rotor layer. In order to specify vector directions it is assumed that: (*a*) the primitive machine is a motor; (*b*) the rotor rotates clockwise; (*c*) negative values of the induced and generated voltages appear in Eqs. (82–85). Suppose (Fig. 2·20*a*) that the rotor is stationary and that the resultant flux due to all windings alternates in time. *Consider first the direction of $\mathbf{\Phi}$ in a single layer winding of the winding of the rotor.* Now a two-pole sinusoidal wave in a winding can be represented by a vector drawn from the axis of the rotor to the positive maximum value of the wave. The maximum induced voltage results in coil *AB* since this loop links the maximum number of lines. Consequently, the vector *OL* (Fig. 2·20*a*) represents $\mathbf{\Phi}$. The directions of $\mathbf{\Phi}$ and $\dfrac{d\mathbf{\Phi}}{dt}$ are the same.

The direction of $\mathbf{\Psi}$ in a single layer winding of the rotor is next considered. Suppose that the rotor moves at speed $p\theta$ and that the flux is stationary and constant in time. The maximum voltage is generated in the coil *CD* (Fig. 2·20*b*). The vector *OF* represents $\mathbf{\Psi}$. The indi-

cated relations (Fig. 2·20*b*) between the direction of Ψ and the direction of the generated voltages in the rotor conductors are correct because the negative of the generated voltages are required for Eqs. (83–84).

It is evident (Figs. 2·20) that the vectors Ψ and Φ_r in a single layer winding are perpendicular. Moreover, their magnitudes are equal. From these two relations Ψ is obtainable from Φ_r. It is evident from Figs. (2·20) that to obtain Ψ from Φ_r it is necessary only to replace the

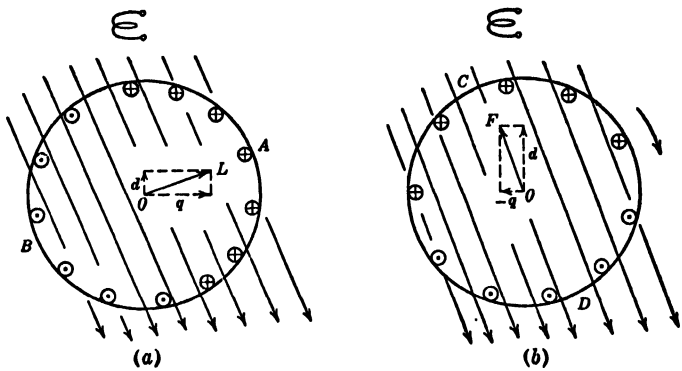

FIG. 2·20. Direction of Waves Inside Machine.
(*a*) Direction of Φ and $d\Phi/dt$.
(*b*) Direction of Ψ and $\Psi\ p\theta$.

q axis of Φ_r by **d** and the **d** axis of Φ_r by $-$**q**. For comparison, in a single layer,

$$\Phi_r = (i^{ds}M_d + i^{dr}L_{dr})\mathbf{d}_r + (i^{qr}L_{qr} + i^{qs}M_q)\mathbf{q}_r, \qquad [100]$$

$$\Psi = (i^{ds}M_d + i^{dr}L_{dr})(-\mathbf{q}) + (i^{qr}L_{qr} + i^{qs}M_q)\mathbf{d}. \qquad [101]$$

The orthogonality of Φ_r and Ψ is verified by $\Phi_r \cdot \Psi = 0$.

2·56. Torque Tensor. The vector Φ is expressible as the product $\mathbf{L}\cdot\mathbf{i}$. It is desirable to express Ψ as a similar product. Accordingly, define **G**, the torque tensor, by the relation

$$\mathbf{G} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & 0 & 0 & 0 & 0 \\ \hline \mathbf{d}_r & 0 & 0 & L_{qr} & M_q \\ \hline \mathbf{q}_r & -M_d & -L_{dr} & 0 & 0 \\ \hline \mathbf{q}_s & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad [102]$$

The above tensor represents the mutual inductances between windings on axis **d** and those on axis **q** due to the existence of rotation. The extension (scarcely a generalization) of this definition to a machine with any number of layers is obvious. (See Ex. 2, problem set XII.) From Eq. (101) it is clear that

$$\boldsymbol{\Psi} = \mathbf{G} \cdot \mathbf{i} \quad \text{and} \quad (p\theta)\boldsymbol{\Psi} = p\theta\, \mathbf{G} \cdot \mathbf{i}. \tag{103}$$

In the general case, when the flux-density wave is not sinusoidal in space, the components of **G** differ from those of **L**. The derivation of Eqs. (82–85) or (86–89) is thus concluded.

2·57. Transient Impedance Tensor. Equation (83) has been written

$$\mathbf{e} = (\mathbf{R} + \mathbf{L}p + p\theta\,\mathbf{G}) \cdot \mathbf{i}. \tag{104}$$

Denote $\mathbf{R} + \mathbf{L}p + p\theta\mathbf{G}$ by $\mathbf{Z}$. The matrix $\mathbf{Z}$ is called the **transient impedance tensor** of the first primitive machine. It is of central importance in the sections which follow. For a two-layer machine it is

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds} + L_{ds}p & M_d p & 0 & 0 \\ \hline \mathbf{d}_r & M_d p & r_r + L_{dr}p & L_{qr}p\theta & M_q p\theta \\ \hline \mathbf{q}_r & -M_d p\theta & -L_{dr}p\theta & r_r + L_{qr}p & M_q p \\ \hline \mathbf{q}_s & 0 & 0 & M_q p & r_{qs} + L_{qs}p \\ \hline \end{array} \tag{105}$$

2·58. Direction of Rotation. Since only the relative rotation of the stator and rotor members determine the equation of voltage, all the above tensors are valid without any change if the salient pole rotates in the opposite direction and the smooth member is stationary, as shown in Fig. 2·201. The reference frames now rotate together with the salient poles, as in a synchronous machine. Instead of stator and rotor subscripts s and r the members may be called field and armature (subscripts f and a). When the direction of rotation of the armature or field changes, then $p\theta$ in $\mathbf{Z}$ changes sign. Otherwise all tensors remain the same.

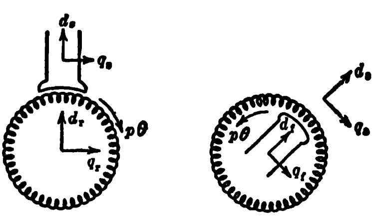

FIG. 2·201. Direction of Rotation.

2·59. Zero-phase-sequence Quantities. When there are zero-phase currents i^{so} and i^{ro} in both stator and rotor windings, as in case of unbalanced three-phase machines, two additional rows and columns exist in **Z**, also in **R**, **L**, and **G**.

Z =

	$\mathbf{d}_s$	$\mathbf{d}_r$	$\mathbf{q}_r$	$\mathbf{q}_s$	$\mathbf{0}_s$	$\mathbf{0}_r$
$\mathbf{d}_s$	$r_{ds}+L_{ds}p$	$M_d p$				
$\mathbf{d}_r$	$M_d p$	$r_r+L_{dr}p$	$L_{qr}p\theta$	$M_q p\theta$		
$\mathbf{q}_r$	$-M_d p\theta$	$-L_{dr}p\theta$	$r_r+L_{qr}p$	$M_q p$		
$\mathbf{q}_s$			$M_q p$	$r_{qs}+L_{qs}p$		
$\mathbf{0}_s$					$r_{so}+L_{so}p$	
$\mathbf{0}_r$						$r_{ro}+L_{ro}p$

EXERCISES XII

1. The notation in Eqs. (90–93) is easily extended to machines having any number of windings on both rotor and stator. For $m = n = 2$ the unit vectors are $\mathbf{d}_{s2}$, $\mathbf{d}_{s1}$, $\mathbf{d}_{r1}$, $\mathbf{d}_{r2}$, $\mathbf{q}_{r2}$, $\mathbf{q}_{r1}$, $\mathbf{q}_{s1}$, $\mathbf{q}_{s2}$. Write out in detail the tensors **e**, **i**, **R**, **L**, $\mathbf{L}_r$, $\mathbf{L}_s$ for a machine for which $m = n = 2$. See Ex. 2 for **G**.

2. For $m = n = 2$ write out the equations corresponding to Eqs. (94–95). For $m = n = 2$, obtain

G =

	$\mathbf{d}_{s2}$	$\mathbf{d}_{s1}$	$\mathbf{d}_{r1}$	$\mathbf{q}_{r1}$	$\mathbf{d}_{r2}$	$\mathbf{q}_{r2}$	$\mathbf{q}_{s1}$	$\mathbf{q}_{s2}$
$\mathbf{d}_{s1}$	0	0	0	0	0	0	0	0
$\mathbf{d}_{s2}$	0	0	0	0	0	0	0	0
$\mathbf{d}_{r1}$	0	0	0	0	M_{qr}	L_{rq1}	M_{q11}	M_{q12}
$\mathbf{d}_{r2}$	0	0	0	0	L_{qr2}	M_{qr}	M_{q21}	M_{q22}
$\mathbf{q}_{r1}$	$-M_{d22}$	$-M_{d21}$	$-M_{dr}$	$-L_{dr2}$	0	0	0	0
$\mathbf{q}_{r2}$	$-M_{d12}$	$-M_{d11}$	$-L_{dr1}$	$-M_{dr}$	0	0	0	0
$\mathbf{q}_{s1}$	0	0	0	0	0	0	0	0
$\mathbf{q}_{s2}$	0	0	0	0	0	0	0	0

(7)

Derived Machines with Stationary Reference Axes (Constant Rotor Speed)

The equations of performance of most derived rotating machines running at constant rotor speed can be set up using stationary reference axes. Exceptions are pointed out in Sec. 9. The equations of performance and their solution for a few derived machines are given in this section.

2·60. Equations of Performance of Derived Machines. In §2·36 the method of obtaining, from the primitive network, the equations of performance of any derived mesh network was explained. It consisted in obtaining necessary relations between n old and s new currents or between m primitive and s derived currents. From these relations the C-transformation tensor was written down.

The procedure in obtaining, from the primitive machine with stationary reference axes, the equations of performance of a derived machine with stationary reference axes is similar. It is necessary to obtain relations between the $m + n$ old currents of the primitive machine and $s = m' + n'$ new currents of the derived machine. From these relations the $\mathbf{C}$ transformation is obtained. Definite rules for the procedure are given in §2·62.

The tensors of the primitive machine with stationary reference axes and constant rotor speed are $\mathbf{i}$, $\mathbf{e}$, $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$, and $\mathbf{Z}$. The corresponding tensors for the derived machines will be denoted by $\mathbf{i}'$, $\mathbf{e}'$, $\mathbf{R}'$, $\mathbf{L}'$, $\mathbf{G}'$, and $\mathbf{Z}'$. By means of the second generalization postulate §2·28 it easily follows that

$$\begin{aligned} &\mathbf{i} = \mathbf{C}\cdot\mathbf{i}', \quad \mathbf{e}' = \mathbf{C}_t\cdot\mathbf{e}, \quad \mathbf{R}' = \mathbf{C}_t\cdot\mathbf{R}\cdot\mathbf{C}, \quad \mathbf{L}' = \mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C}, \\ &\mathbf{G}' = \mathbf{C}_t\cdot\mathbf{G}\cdot\mathbf{C}, \quad \mathbf{Z}' = \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C}. \end{aligned} \qquad [106]$$

The currents are found by $\mathbf{i}' = \mathbf{Z}^{-1}\cdot\mathbf{e}'$; the torque by $f' = \mathbf{i}'\cdot\mathbf{G}'\cdot\mathbf{i}'$.

Before stating general rules §2·62 for the determination of $\mathbf{C}$ for derived machines the equations of performance of the repulsion motor will be derived in detail.

2·61. Introductory Example; Single-phase Repulsion Motor. Both the derivation and solution (for stationary axes) of the equation of performance are routine processes. Moreover, these routine processes are the same for all derived machines with stationary reference axes. The analyses of types of machines which differ as much among themselves as a compound direct-current motor, an alternator, a double squirrel-

cage induction motor, or a repulsion motor is substantially one analysis. For this reason a general idea of the routine process of obtaining and solving the equations of performance of derived machines can be obtained by the application of the tensor theory to a specific example. We shall employ a single-phase repulsion motor.

It may be stated to the mathematics student that a single-phase repulsion motor is one of the simplest alternating-current motors. The field or stator winding consists of a single layer which is supplied by single phase alternating-current voltage. The rotor winding is also a single layer winding. It is a commutated drum-armature winding similar to that of a direct-current motor. In Fig. 2·21*a*, $C'D'$ represents the plane of the stator coil; CD is the line of the brush axis inclined at an angle to $C'D'$. The brushes at C and D are externally short-circuited.

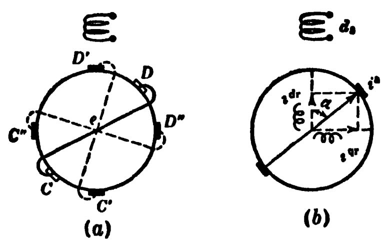

FIG. 2·21. Repulsion Motor.

That a torque is exerted on the rotor for $0 < \alpha < 90°$ can be seen as follows. If $\alpha = 0$ so that the brush axis is in line with the field poles, then currents are induced in the coil $C'eD'$. The torques exerted in both directions of possible rotation are numerically equal but oppositely directed and the resultant torque is zero. If $\alpha = 90°$ no currents are induced in $C''eD''$. Finally, if α has any intermediate value between 0° and 90° then the planes of the stator and short-circuited rotor coil intersect in the rotor axis. A force of repulsion acting in the direction of the displacement exists between the two coils. This is in accordance with the experimental fact that conductors carrying currents in opposite directions repel each other. This repulsion furnishes the operating torque.

(*a*) *The* **C** *transformation.* To obtain **C** for the repulsion motor it is necessary only to compare the windings described above (Fig. 2·21*b*) with the windings of the first primitive machine (Fig. 2·18*b*). Evidently, the relations between the primitive currents **i** and the derived currents **i′** are (from Fig. 2·21*b*)

$$
\begin{aligned}
i^{ds} &= \quad i^{ds} + 0\cdot i^{a} \\
i^{dr} &= 0\cdot i^{ds} + (\cos\alpha)\, i^{a} \quad \text{whence} \quad \mathbf{C} = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & 1 & 0 \\ \hline \mathbf{d}_r & 0 & \cos\alpha \\ \hline \mathbf{q}_r & 0 & \sin\alpha \\ \hline \end{array} \qquad [107] \\
i^{qr} &= 0\cdot i^{ds} + (\sin\alpha)\, i^{a}
\end{aligned}
$$

(*b*) *Primitive and derived tensors.* By comparison of the winding layers of the derived machine (Fig. 2·21*b*) with those of the primitive machine (Fig. 2·18*b*) the **Z** transient matrix (Eq. 105) is found to be

$$\begin{array}{c|c|c|c|}
 & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r \\
\hline
\mathbf{d}_s & r_s + L_{ds}p & M_d p & 0 \\
\hline
\mathbf{d}_r & M_d p & r_r + L_{dr}p & L_{qr}p\theta \\
\hline
\mathbf{q}_r & -M_d p\theta & -L_{dr}p\theta & r_r + L_{qr}p \\
\hline
\end{array}
=
\begin{array}{c|c|c|c|}
 & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r \\
\hline
\mathbf{d}_s & r_s + L_s p & Mp & 0 \\
\hline
\mathbf{d}_r & Mp & r_r + L_r p & L_r p\theta \\
\hline
\mathbf{q}_r & -Mp\theta & -L_r p\theta & r_r + L_r p \\
\hline
\end{array}
\qquad [108]$$

The equality sign in (108) is justified by the fact that the machine has a smooth air gap resulting in $L_{dr} = L_{qr} = L_r$.

The transient impedance matrix of the derived machine is

$$\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C} =
\begin{array}{c|c|c|}
 & \mathbf{d}_s & \mathbf{a} \\
\hline
\mathbf{d}_s & r_s + L_s p & M(\cos\alpha)p \\
\hline
\mathbf{a} & M[(\cos\alpha)p - (\sin\alpha)p\theta] & r_r + L_r p \\
\hline
\end{array}
\qquad [109]$$

The induced metric tensor of the derived machine is

$$\mathbf{L}' = \mathbf{C}_t \cdot \mathbf{L} \cdot \mathbf{C} =
\begin{array}{c|c|c|}
 & \mathbf{d}_s & \mathbf{a} \\
\hline
\mathbf{d}_s & L_s & M\cos\alpha \\
\hline
\mathbf{a} & M\cos\alpha & L_r \\
\hline
\end{array}
\qquad [110]$$

The voltage vector of the primitive machine is

$$\mathbf{e} =
\begin{array}{|c|c|c|}
\mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r \\
\hline
e_s & 0 & 0 \\
\hline
\end{array}
\qquad [111]$$

The voltage vector of the derived machine is

$$\mathbf{e}' = \mathbf{C}_t \cdot \mathbf{e} =
\begin{array}{|c|c|}
\mathbf{d}_s & \mathbf{a} \\
\hline
e_s & 0 \\
\hline
\end{array}
\qquad [112]$$

The admittance matrix of the derived machine is

$$\mathbf{Y}' = \mathbf{Z}'^{-1} = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & (r_r + L_r p)1/D & (-M\cos\alpha)p\,1/D \\ \hline \mathbf{a} & -M[(\cos\alpha)p - (\sin\alpha)p\theta]1/D & (r_s + L_s p)1/D \\ \hline \end{array} \qquad [113]$$

where

$$D = (L_s L_r - M^2\cos^2\alpha)p^2 + (r_r L_s + r_s L_r + M^2 \sin\alpha\cos\alpha\, p\theta)p + r_r r_s.$$

(c) *Equation of performance.* The equation of performance is

$$\mathbf{Z}'\cdot\mathbf{i}' = \mathbf{e}' \quad \text{or} \quad \mathbf{i}' = \mathbf{Y}'\cdot\mathbf{e}'.$$

The symbolic solution of the last equation for $\mathbf{i}'$ is

$$\mathbf{i}' = \mathbf{Z}'^{-1}\cdot\mathbf{e}' = \mathbf{Y}'\cdot\mathbf{e}' = \frac{(r_r + L_r p)}{D}\, e_d\mathbf{d}_s + \frac{M[(\sin\alpha)p\theta - (\cos\alpha)p]}{D}\, e_d\mathbf{a} \qquad [114]$$

where

$$D = (L_s L_r - M^2\cos^2\alpha)p^2 + (r_r L_s + r_s L_r + M^2 \sin\alpha\cos\alpha\, p\theta)p + r_r r_s.$$

(d) *Transient current solution.* Equation (114) is the symbolic current solution under all conditions. The transient solution due to suddenly impressed constant voltage can be obtained by the Heaviside operational calculus as if the network were a stationary linear network. *This startling fact is true not only for the repulsion motor, but for all rotating machines with stationary reference axes and constant rotor speed.*

Under constant rotor speed $p\theta = v\omega$ where ω is constant synchronous speed and v is a proper fraction. To obtain the transient currents replace, in Eq. (114), e_d by $\mathit{1}e_d$ where $\mathit{1}$ is the Heaviside unit function, and substitute the symbolic expression for the current in the Bromwich line integral.[18] The substitution yields

$$\mathbf{i}' = \frac{1}{2\pi j}\left\{\frac{A}{D_0}\, e_d\mathbf{d}_s \int_0 \frac{e^{\lambda t}d\lambda}{(\lambda+\alpha)^2+\beta^2} + \frac{E}{D_0}\, e_d\mathbf{d}_s \int_0 \frac{e^{\lambda t}\,d\lambda}{\lambda[(\lambda+\alpha)^2+\beta^2]}\right.$$
$$\left. + e_d\mathbf{a}\left[\frac{B}{D_0}\int_0 \frac{e^{\lambda t}d\lambda}{\lambda[(\lambda+\alpha)^2+\beta^2]} - \frac{C}{D_0}\int_0 \frac{e^{\lambda t}d\lambda}{(\lambda+\alpha)^2+\beta^2}\right]\right\} \qquad [115]$$

where A, B, C, E, D_0, α, and β are constants.

(e) *Steady-state current solution.* The steady-state solution for terminal voltage $(e \sin \omega t)\mathbf{d}_s$ is obtained as in the case of linear stationary networks.

[18] Volume I, p. 262.

As a general procedure convenient for all rotating machines it is systematic to write the steady-state equation of performance for the machine in question. This equation is $\mathbf{Z}_s \cdot \mathbf{i}_s = \mathbf{e}_s$ where $\mathbf{Z}_s$ is the steady-state impedance tensor of the derived machine obtained from $\mathbf{Z}$ by the substitution $p = j\omega$, $p\theta = \omega v$, $\omega L = X$, and $\omega M = X_m$. The impedance matrix $\mathbf{Z}'_s$ for the repulsion motor is

$$\begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & r_s + jX_s & jX_m \cos \alpha \\ \hline \mathbf{a} & X_m(j \cos \alpha - v \sin \alpha) & r_r + jX_r \\ \hline \end{array} \qquad [116]$$

The steady-state admittance $\mathbf{Y}'_s$ of the repulsion motor is

$$\mathbf{Y}'_s = \begin{array}{c|c|c|} & \mathbf{d}_s & \mathbf{a} \\ \hline \mathbf{d}_s & (r_r + jX_r)1/D_s & -jX_m \cos \alpha 1\ /D_s \\ \hline \mathbf{a} & X_m(\sin \alpha\, v - j \cos \alpha)1/D_s & (r_s + jX_s)1/D_s \\ \hline \end{array} \qquad [117]$$

where

$$D_s = (r_r r_s + X_m^2 \cos^2 \alpha - X_s X_r) + j(r_r X_s + r_s X_r + v\, X_m^2 \sin \alpha \cos \alpha).$$

The steady-state currents are

$$\mathbf{i}'_s = \mathbf{Y}'_s \cdot \mathbf{e}' = \hat{e}\, \frac{(r_r + jX_r)\mathbf{d}_s}{D_s} + \hat{e} X_m \frac{(\sin \alpha\, v - j \cos \alpha)\mathbf{a}}{D_s}. \qquad [118]$$

(*f*) *Transient and steady-state currents.* If the suddenly impressed voltage is $(e_d \sin \omega t)\mathbf{d}_s$ then both the transient and steady-state currents are obtainable by the substitution of the results of Eq. (115) in Duhamel's superposition formula.[19]

(*g*) *Transient torque.* The torque tensor $\mathbf{G}'$ for the derived machine is

$$\mathbf{G}' = \mathbf{C}_t \cdot \mathbf{G} \cdot \mathbf{C}. \qquad [119]$$

The transient torque f_t is given by

$$f_t = \mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}' \qquad [120]$$

where the instantaneous current $\mathbf{i}'$ is given in Eq. (115).

[19] E. J. Berg, *Heaviside's Operational Calculus*, p. 67; V. Bush, *Operational Circuit Analysis*, p. 56.

The torque tensor for the repulsion motor is

$$\mathbf{G}' = \mathbf{C}_t \cdot \mathbf{G} \cdot \mathbf{C} = \begin{array}{c|c|} & \mathbf{d}_s \\ \cline{2-2} \mathbf{a} & -M \sin \alpha \\ \cline{2-2} \end{array} \qquad [121]$$

(*h*) *Transient and steady-state torque.* The total torque is given by

$$f_{ts} = \mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}' \qquad [122]$$

where $\mathbf{i}'$ is the current of heading (*f*) above. The result will contain (1) transient torque, (2) steady part of the steady-state torque, (3) oscillating component of steady-state torque. Important torque calculations are, however, more easily carried out as shown in headings (*i*) and (*j*) following.

(*i*) *Steady part of steady-state torque.* The steady part of the steady-state torque is given by taking the real part of

$$f_{sp} = \mathbf{i}'^* \cdot \mathbf{G}' \cdot \mathbf{i}' \qquad [123]$$

where $\mathbf{i}'^*$ is the complex conjugate of $\mathbf{i}'$ as given in Eq. (118). The quantity f_{sp} for the repulsion motor is

$$f_{sp} = \hat{e}\, \frac{X_m(\sin \alpha\, v + j \cos \alpha)}{D_s^*} \left(-X_m \sin \alpha\right) \frac{\hat{e}(r_r + jX_r)}{D_s}.$$

When the torque is computed in synchronous watts the torque tensor is to be multiplied by ω. The ω has already been multiplied into $\mathbf{G}'$ in the expression above since ωM has been replaced by X_m. The real part of f_{sp} is easily written out.

(*j*) *Oscillating component of steady-state torque.* The oscillating component of the steady-state torque is the non-transient and non-steady part of the torque in heading (*h*).

EXERCISES XIII

1. The winding diagram of a single-phase induction motor is shown in Fig. 2·22. It will be shown in §2·62 on the C tensor that the C tensor for a single-phase induction motor with stationary reference axes is

$$\mathbf{C} = \begin{array}{c|c|c|c|} & \mathbf{d}_s & \mathbf{d}_r & \mathbf{q}_r \\ \hline \mathbf{d}_s & 1 & 0 & 0 \\ \hline \mathbf{d}_r & 0 & 1 & 0 \\ \hline \mathbf{q}_r & 0 & 0 & 1 \\ \hline \end{array}$$

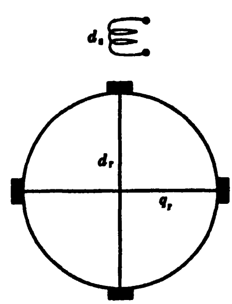

FIG. 2·22. Single-phase Induction Motor.

Compare the windings indicated in Fig. 2·22 with those of the primitive machine in Fig. 2·18*b* and by inspection fill in the transient impedance matrix, Eq. (105). Compute $\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$. Carry out headings (*b*) and (*c*) of §2·61 and obtain the equation of performance corresponding to Eq. (114).

2. The winding diagram of a compound direct-current motor is shown in Fig. 2·23. As shown later the C-transformation tensor (or transformation matrix which is one manifestation of the transformation tensor) is

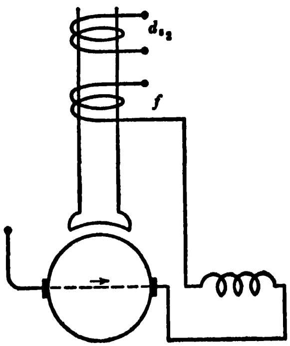

FIG. 2·23. Compound Direct-current Motor.

$$\mathbf{C} = \begin{array}{c|cc} & \mathbf{d}_{s2} & \mathbf{f} \\ \hline \mathbf{d}_{s2} & 1 & 0 \\ \mathbf{d}_{s1} & 0 & -n_d \\ \mathbf{q}_{r1} & 0 & 1 \\ \mathbf{q}_{s1} & 0 & -n_q \end{array}$$

Compare the windings indicated in Fig. 2·23 with those of the primitive machine of Fig. 2·18*b* and by inspection fill in the transient impedance matrix, Eq. (105). Compute $\mathbf{Z}' = \mathbf{C}_t \cdot \mathbf{Z} \cdot \mathbf{C}$. Carry out headings (*b*) and (*c*) of §2·61 and obtain the equation of performance corresponding to Eq. (114).

2·62. The C-Transformation Tensor for Machines with Stationary Reference Axes. The construction of the **C**-transformation matrix (one manifestation of the **C** tensor) can be broken up into three steps.

(*a*) *Winding diagram.* The first step is a sketch of the winding diagram of the derived machine in question. Such a diagram for a single-phase induction motor occurs in Fig. 2·22. The diagram must show the (1) number of layers of windings on the stator and their relative positions, (2) number of layers of windings on the rotor and their relative positions, (3) electrical connections between the different rotor windings and between the rotor and stator windings, (4) physical or fictitious brushes. (See §2·52.)

(*b*) *Comparison of winding diagrams.* The second step is a comparison of the winding diagram of the particular derived machine with the winding diagram of the primitive machine shown in Fig. 2·18*b*. The currents of the primitive machine are denoted by unprimed letters, i^{ds2}, i^{ds1}, i^{dr1}, i^{dr2}, etc. The currents (really components of the current vector) of the derived machine are denoted by primed letters, $i^{a'}$, $i^{b'}$, $i^{c'}$, etc. (See Eq. 107.)

(*c*) *Current relations: equations.* The third step consists in expressing the old or primitive currents in terms of the new or derived currents

of the derived machine. In general, the number of derived currents is smaller than the number of primitive currents. The primitive currents are written on the left-hand side of the equal signs; the derived currents on the right. The **C** matrix is the matrix of the coefficients of the derived currents. (See Eq. 107 and the **C** matrix for the single-phase repulsion motor.) In writing the current equations four important principles are employed.

(1) If a winding of the derived machine is in the same position as in the primitive machine, then $i^d = i^{d'}$ (see Fig. 2·22 and the **C** matrix). Let the magnetomotive force of some winding whose current is, say $i^{d'}$, be taken as standard. If the number of turns in the winding i^d is n times that of $i^{d'}$, then $i^d = n\, i^{d'}$. (See Fig. 2·23 and the adjacent **C** matrix.)

(2) If two windings are connected in series their currents are replaced by one current and the number of columns in **C** diminishes. The second half of principle (1) may necessarily be employed. (See Fig. 2·23.)

(3) If a set of brushes on a rotor winding is shifted through an angle α the winding in the brush axis can be considered to lie on the original layer of winding before the shift took place. This is because the rotor windings are symmetrically distributed. If $i^{f'}$ is the current in the brush axis the relations between $i^{f'}$ and the primitive currents i^{dr} and i^{qr} are

$$i^{dr} = i^{f'} \cos \alpha$$

$$i^{qr} = i^{f'} \sin \alpha.$$

If there are two sets of brushes and the first is shifted through the angle α, the second through the angle β, then the relations between the primitive and derived rotor currents are

$$i^{dr} = i^{f'} \cos \alpha - i^{g'} \sin \beta$$

$$i^{qr} = i^{f'} \sin \alpha + i^{g'} \cos \beta.$$

For examples of single and double sets of brushes, see Eq. (106) and the **C** matrix belonging to Fig. 2·27 respectively.

(4) If a winding on the stator is shifted through an angle α then it is assumed in the analysis that it lies on a different layer from the other stator windings. (See Fig. 2·26 and the **C** matrix.)

The four principles above pertain to the machines with stationary reference axes. Additional principles for machines with moving reference axes are given in Secs. 8–9.

2·63. Performance Calculations for Machines with Stationary Axes. In §2·61 are the performance calculations of the single-phase repulsion motor. They consist of the ten steps (*a*), (*b*), ···, (*j*) of §2·61.

The performance calculations for each type of derived machine, possessing stationary reference axes and constant rotor speed, are identical; only the tensors and the C-transformation matrix differ. Most machines can be analyzed by means of stationary reference axes under the conditions given in the last sentence. (See Sec. 8 for moving axes.) The equation of performance is linear with constant coefficients and can be solved in terms of time functions either by means of the Heaviside operational calculus or by the principles of §§1·26–1·29. These two characteristics of only a very small part of Kron's analysis of rotating electrical machinery are alone sufficient to rank it as an important achievement.

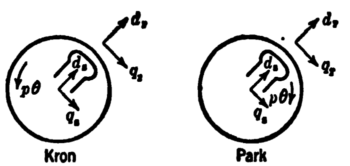

FIG. 2·231. Sign Conventions.

2·64. Sign Conventions and Machine Constants. In analyzing the synchronous generator, Park assumed a sign convention which differs from that used here. In this chapter the sign conventions are the same as those used by induction motor engineers and which also follow from the dynamical equations of Lagrange. Park's sign convention differs in two respects: (*a*) Assuming that the salient pole of the primitive machine rotates (§2·50) and the armature is stationary, in this chapter the salient pole rotates from d to $-q$ though with Park it rotates from d to q. Hence, to check Park's results all $p\theta$ of the present chapter should be replaced by $-p\theta$. (*b*) In this chapter every term in the equation of voltage represents an impressed voltage, although Park uses generated voltages which are the negative of the impressed voltages.

$$\text{(Kron)}\quad \mathbf{e}_{\text{imp}} = \mathbf{Z}\cdot\mathbf{i}, \qquad \text{(Park)}\quad \mathbf{e}_{\text{gen}} = -\mathbf{Z}\cdot\mathbf{i},$$

i.e., assuming zero speed for a single coil

$$\text{(Kron)}\quad \mathbf{e}_{\text{imp}} = Ri + L\frac{di}{dt}, \qquad \text{(Park)}\quad \mathbf{e}_{\text{gen}} = -Ri - L\frac{di}{dt}.$$

The convention of Park differs also from that universally used in stationary network analysis.

In addition to sign conventions, Park also differs in symbolism from that of this chapter in two respects: (*a*) Park uses a per unit system, hence among others he denotes inductances L by X (since numerically a reactance is equal to an inductance in the per unit system). (*b*) Park assumes the field winding and the amortisseur winding to be permanently short-circuited, so that the remaining equations contain short-circuited inductances $X(p)$ instead of open-circuited inductances.

Of course, by eliminating the variables of the same two windings by the method of problem 3, problem set XVII, the equations of this chapter are reducible to Park's results.

EXERCISES XIV

The following exercises pertain to machines having stationary reference axes and constant rotor speed.

1. Show that the winding diagram for the salient-pole synchronous machine is that shown in Fig. 2·24. Show that

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \mathbf{d}_{s2} & \mathbf{d}_{s1} & \mathbf{d}_r & \mathbf{q}_r & \mathbf{q}_s \end{array} \\
\begin{array}{r} \mathbf{d}_{s2} \\ \mathbf{d}_{s1} \\ \mathbf{C} = \mathbf{d}_r \\ \mathbf{q}_r \\ \mathbf{q}_s \end{array} &
\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

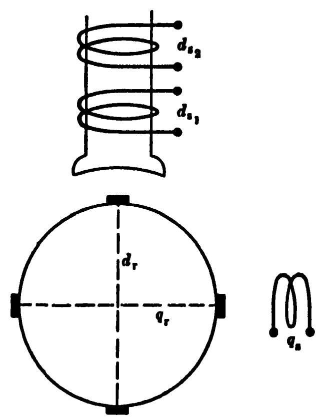

FIG. 2·24. Salient-pole Synchronous Machine.

2. Show that in a synchronous motor running at synchronous speed and under balanced conditions the applied voltages are all constant.

3. Obtain the equation of performance (Eq. 114) for the synchronous machine of Ex. 1.

4. Obtain, from the equation of performance, the symbolic solution for i^{dr} and i^{qr} in Ex. 1.

5. Express i^{dr} and i^{qr} in Ex. 4 as functions of the time by Heaviside's operational calculus.

6. Compute $\mathbf{G}' = \mathbf{C}_t \cdot \mathbf{G} \cdot \mathbf{C}$ for the salient-pole synchronous machine.

7. Compute the steady-state torque of the salient-pole synchronous machine. Obtain expressions for the transient torque, under three-phase short-circuit, of a synchronous machine. Compare with R. E. Doherty and C. A. Nickle, "Three-Phase Short-Circuit of Synchronous Machines," *Trans. A.I.E.E.*, 49, April, 1930.

8. Show that the winding diagram for the double squirrel-cage induction motor is that shown in Fig. 2·25. Show that

$$C = \begin{array}{c|cccccc|} & \mathbf{d}_s & \mathbf{d}_{r1} & \mathbf{d}_{r2} & \mathbf{q}_{r2} & \mathbf{q}_{r1} & \mathbf{q}_s \\ \hline \mathbf{d}_{s2} & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{d}_{r1} & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathbf{d}_{r2} & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{q}_{r2} & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{q}_{r1} & 0 & 0 & 0 & 0 & 1 & 0 \\ \mathbf{q}_{s1} & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

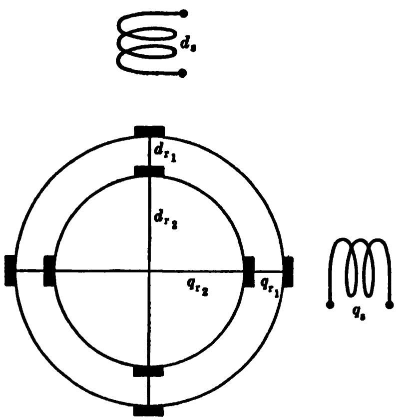

FIG. 2·25. Double Squirrel-cage Induction Motor.

9. Show that for the double squirrel-cage induction motor running in steady-state operation all applied voltages are sinusoidal.

10. Obtain the equation of performance (Eq. 114) for the double squirrel-cage motor.

11. Solve Exs. 7, 8, 9 for the asymmetrical squirrel-cage induction motor.

12. Obtain the symbolic solution for i^{dr} and i^{qr} of the asymmetrical induction motor.

13. By means of the Heaviside operational calculus solve for i^{dr} and i^{qr} in Ex. 12. The applied voltages are sinusoidal.

14. Solve for steady-state i^{dr} and i^{qr} in Ex. 12. The applied voltages are sinusoidal.

15. Obtain the torque tensor for the asymmetrical induction motor.

16. Compute the steady-state torque of the machine in Ex. 11.

17. Obtain the symbolic solution for i^f and i^{ds_2} for the machine of Ex. 2, problem set XIII.

18. Obtain by means of the Heaviside operational calculus transient i^f and $i^{d_{s2}}$.

19 Show that the winding diagram for a shaded-pole motor is that shown in Fig. 2·26. Show that

$$
\mathbf{C} = \begin{array}{c} \\ \mathbf{d}_s \\ \mathbf{d}_{s1} \\ \mathbf{d}_r \\ \mathbf{q}_r \\ \mathbf{q}_s \end{array}
\begin{array}{c}
\begin{array}{cccc} \mathbf{d}_{s2} & \mathbf{a} & \mathbf{d}_r & \mathbf{q}_r \end{array} \\
\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos\alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & \sin\alpha & 0 & 0
\end{array}\right]
\end{array}
$$

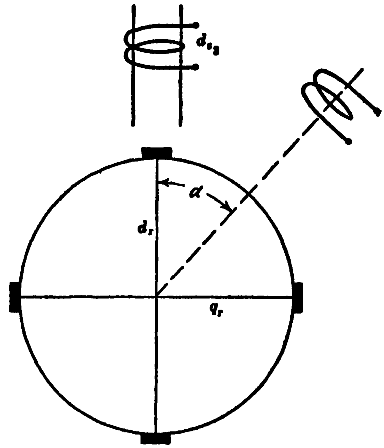

FIG. 2·26. Shaded-pole Motor.

20. Show that the winding diagram for the shunt polyphase commutator motor is that shown in Fig. 2·27. Show that

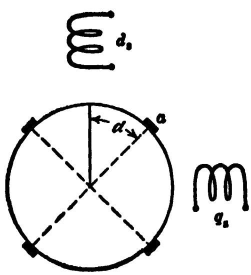

FIG. 2·27. Shunt-polyphase Commutator Motor.

$$
\mathbf{C} = \begin{array}{c} \\ \mathbf{d}_s \\ \mathbf{d}_r \\ \mathbf{q}_r \\ \mathbf{q}_s \end{array}
\begin{array}{c}
\begin{array}{cccc} \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \end{array} \\
\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos\alpha & -\sin\alpha & 0 \\
0 & \sin\alpha & \cos\alpha & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

PROBLEMS XV

1. Derive the equations of performance of the single-phase induction motor.
2. Sketch the winding diagram and obtain the **C** matrix for the Schrage motor. Engineering reference for description of winding layers of the machine, A. S. Langsdorf, *Theory of Alternating-Current Machinery*, p. 752.
3. Sketch the winding diagram and obtain the **C** matrix for the Deri motor.
4. Sketch the winding diagram and obtain the **C** matrix for the phase advancer.
5. Sketch the winding diagram and obtain the **C** matrix for a frequency converter.

(8)

Primitive Machine with Rotating Reference Axes

In this section the equations of performance of machines with rotating rotor axes are derived. Rotating reference axes are necessary for machines possessing only one slip-ring (such as a single-phase alternator) and for machines possessing slip-rings across which the load is unbalanced (for example, the single-phase short circuit of an alternator

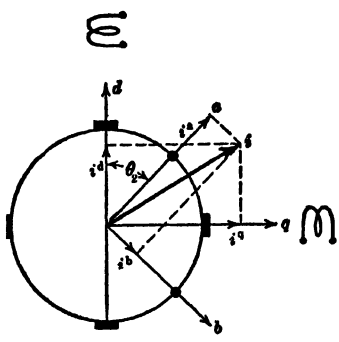

FIG. 2·28. Rotor Reference Axes Rotating at Same Instantaneous Velocity as the Rotor.

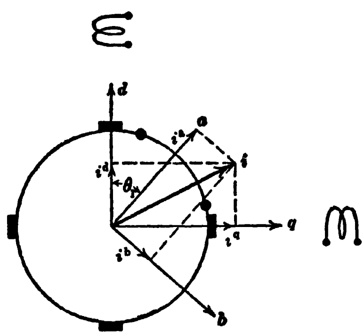

FIG. 2·29. Rotor Reference Axes Rotating at Instantaneous Velocity Different from that of the Rotor.

and for balanced polyphase machines where axes may be assumed rotating with the revolving field, thereby reducing the analysis to that of a direct-current machine (important in hunting studies). For these cases stationary reference axes on the rotor cannot be employed.

2·65. Second Primitive Machine. The second primitive machine can be described by contrast and comparison with the first primitive machine.

In the first primitive machine all reference axes are stationary in space. In the second primitive machine the stator reference axes are stationary axes, but the reference axes on the rotor move with the rotor conductors. (See Fig. 2·28.)

The components of the metric tensor **L** for the first primitive machine are constants. The components of the metric tensor for the second primitive machine are functions of rotor position, i.e., of the time. (For example, see Eq. 126.)

Items (*a*), (*b*), (*c*), (*d*), (*e*), and (*h*) of §2·49 are identical for both machines.

2·66. Starting Points in Deriving the Voltage Equation of the Second Primitive Machine. The voltage equation of the second primitive machine can be derived in at least three different ways. These are characterized by the starting points or fundamental underlying equations. The underlying equations are: (*a*) holonomic equations of Lagrange, (*b*) Maxwell's voltage and torque equations, (*c*) equations of the non-holonomic machine, §2·53. Although only the last method is employed in this chapter, it is instructive to sketch briefly the first two methods.

(*a*) *Lagrange's equations.* (See Sec. 3, Chap. I.) The holonomic equations of Lagrange can be used as a starting point for the equations of the second primitive machine if the rotor reference axes move with the rotor conductors. The instantaneous stored kinetic energy, the dissipation function, and the potential energy are respectively $T = 1/2\, L_{mn}\, i^m i^n$, $F = 1/2\, R_{mn}\, i^m i^n$, and zero. Let x^k denote the total number of charges that have passed through any winding and the angle described by the rotor during some definite time interval. Then $dx^k/dt = i^k$. The voltages applied to any winding and the instantaneous applied shaft torque are denoted by e_k. By substitution in the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^k}\right) - \frac{\partial T}{\partial x^k} + \frac{\partial F}{\partial \dot{x}^k} = e_k$$

and the performance of certain simplifications, the equation of motion of the second primitive machine is

$$e_k = R_{mk} i^m + L_{mk} \frac{di^m}{dt} + [mn, k] i^m i^n \qquad [124]$$

where $[mn, k] = \frac{1}{2}\left(\frac{\partial L_{mk}}{\partial x^n} + \frac{\partial L_{nk}}{\partial x^m} - \frac{\partial L_{mn}}{\partial x^k}\right)$. The geometric object of rank 3, $[mn, k]$ is the holonomic Christoffel symbol of the first kind. The manipulations described above in obtaining Eq. (124) from Lagrange's equations are left as an exercise.

Much analysis remains in constructing the forms of L_{mn} and R_{mn} adaptable to rotating electrical machines and in deriving **C** transformations for winding connections.

(*b*) *Maxwell's voltage and torque equations.* Maxwell's equation of voltage for a system of conductors is

$$\mathbf{e} = \mathbf{R} \cdot \mathbf{i} + \frac{d\mathbf{\Phi}}{dt} \quad \text{or} \quad e_m = R_{mn}\, i^n + \frac{d\varphi_m}{dt}, \qquad [125]$$

where $\mathbf{\Phi} = \mathbf{L}\cdot\mathbf{i}$ is the flux-linkage vector. When $\mathbf{L}$ as a function of angular position of the rotor and $\mathbf{R}$ are known, the voltage equation of a machine can be established. Its equation of torque is

$$f = \frac{\partial T}{\partial \theta} = \frac{1}{2}\mathbf{i}\cdot\frac{\partial \mathbf{L}}{\partial \theta}\cdot\mathbf{i}$$

or

$$f = \frac{1}{2}\frac{\partial L_{mn}}{\partial \theta} i^m i^n.$$

As an example consider the two-phase salient-pole alternator. The components of the inductance tensor $\mathbf{L}$ are found by test by measuring the self- and mutual inductances of the field and armature as a function of the angular position of the rotor relative to the stator. For this machine $\mathbf{L}$ explicitly is

$$\begin{array}{c|cccc|}
 & \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \\
\hline
\mathbf{d}_s & L_{ds} & M_d \cos\theta & -M_d \sin\theta & 0 \\
\mathbf{a} & M_d \cos\theta & L_S + L_D \cos 2\theta & -L_D \sin 2\theta & M_q \sin\theta \\
\mathbf{b} & -M_d \sin\theta & -L_D \sin 2\theta & L_S - L_D \cos 2\theta & M_q \cos\theta \\
\mathbf{q}_s & 0 & M_q \sin\theta & M_q \cos\theta & L_{qs} \\
\hline
\end{array} \qquad [126]$$

where $L_S = (L_{dr} + L_{qr})/2$ and $L_D = (L_{dr} - L_{qr})/2$. The tensor $\mathbf{R}$ has the same form as in the first primitive machine.

It is possible to start with either of the equations of voltage (124) or (125) and to obtain Eq. (83) which is the equation of voltage of the first primitive machine. This is accomplished by changes of variables by means of quasi-holonomic relations between the currents of the second and the currents of the first primitive machine.

Since Eq. (83) can be derived [20] from Eq. (124) or (125), it is reasonable to suppose that the equation of voltage for the second primitive machine can be obtained from Eq. (83). This supposition is correct. We shall follow this method in §2·67.

2·67. Equation of Voltage of Machines with Axes Rotating at Any Speed. Let the reference axes on the stator or rotor of the first primitive machine (Fig. 2·29) be rotating with any instantaneous velocity

[20] See Ex. 1, problem set XVI.

$p\theta_1$ different from the instantaneous velocity of the rotor $p\theta_2$. The equation of voltage for the first primitive machine is

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}p\cdot\mathbf{i} + p\theta_2\mathbf{G}\cdot\mathbf{i} \quad \text{or} \quad e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + p\theta_2 G_n i^n. \quad [127]$$

Let the currents of the new primitive machine be denoted by $\mathbf{i}'$ and let the relation between $\mathbf{i}$ and $\mathbf{i}'$ be given by $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ where $\mathbf{C}$ is a function of θ_2 and where $\mathbf{C}$ is such that the power input $\mathbf{i}\cdot\mathbf{e}$ is invariant. The substitution of $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ in Eq. (127) yields

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{C}\cdot\mathbf{i}' + \mathbf{L}\cdot p(\mathbf{C}\cdot\mathbf{i}') + p\theta_2\,\mathbf{G}\cdot\mathbf{C}\cdot\mathbf{i}'$$

or

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{C}\cdot\mathbf{i}' + \mathbf{L}\cdot\frac{d\mathbf{C}}{dt}\cdot\mathbf{i}' + \mathbf{L}\cdot\mathbf{C}\cdot\frac{d\mathbf{i}'}{dt} + p\theta_2\,\mathbf{G}\cdot\mathbf{C}\cdot\mathbf{i}'.$$

The multiplication of the last equation by $\mathbf{C}_t$ gives

$$\mathbf{C}_t\cdot\mathbf{e} = \mathbf{C}_t\cdot\mathbf{R}\cdot\mathbf{C}\cdot\mathbf{i}' + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}\frac{\partial\theta_1}{\partial t}\cdot\mathbf{i}' + \mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C}\cdot\frac{d\mathbf{i}'}{dt} + p\theta_2\,\mathbf{C}_t\cdot\mathbf{G}\cdot\mathbf{C}\cdot\mathbf{i}'$$

or

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot\frac{d\mathbf{i}'}{dt} + p\theta_2\,\mathbf{G}'\cdot\mathbf{i}' + p\theta_1\,\mathbf{V}'\cdot\mathbf{i}' \qquad [128]$$

where

$$\begin{aligned}\mathbf{e}' &= \mathbf{C}_t\cdot\mathbf{e}\\ \mathbf{R}' &= \mathbf{C}_t\cdot\mathbf{R}\cdot\mathbf{C}\\ \mathbf{L}' &= \mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C} \qquad [129]\\ \mathbf{G}' &= \mathbf{C}_t\cdot\mathbf{G}\cdot\mathbf{C}\\ \mathbf{V}' &= \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}.\end{aligned}$$

Equation (128) is the voltage equation for machines with axes rotating with a velocity different from that of the rotor. The matrix $\mathbf{V}$ is called the **Christoffel object.**

To obtain the transformation formula for $\mathbf{V}$, let Eq. (127) be written

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + \mathbf{L}p\cdot\mathbf{i} + p\theta_2\,\mathbf{G}\cdot\mathbf{i} + p\theta_1\,\mathbf{V}\cdot\mathbf{i}$$

where $\mathbf{V} = \mathbf{O}$. Making the substitution $\mathbf{i} = \mathbf{C}\cdot\mathbf{i}'$ in this equation, mul-

tiplying through by $\mathbf{C}_t$, and using Eqs. (129), we have

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + p\theta_2\mathbf{G}'\cdot\mathbf{i}' + p\theta_1\left(\mathbf{C}_t\cdot\mathbf{V}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}\right)\cdot\mathbf{i}'$$

or

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'\cdot p\mathbf{i}' + p\theta_2\mathbf{G}'\cdot\mathbf{i}' + p\theta_1\mathbf{V}'\cdot\mathbf{i}' \tag{130}$$

where

$$\mathbf{V}' = \mathbf{C}_t\cdot\mathbf{V}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}. \tag{131}$$

Equation (131) is the transformation formula for **V**. Evidently **V** is not a tensor.

If $p\theta_1$ and $p\theta_2$ are constants, then Eq. (130) can be written

$$\mathbf{e}' = (\mathbf{R}' + \mathbf{L}'p + p\theta_2\,\mathbf{G}' + p\theta_1\,\mathbf{V}')\cdot\mathbf{i}' = \mathbf{Z}'\cdot\mathbf{i}'$$

where

$$\mathbf{Z}' = \mathbf{R}' + \mathbf{L}'p + p\theta_2\,\mathbf{G}' + p\theta_1\,\mathbf{V}'.$$

2·68. Equation of Voltage for the Second Primitive Machine (or for the Machine with Reference Axes Attached to the Rotor). If the reference axes rotate at the same velocity as the rotor then $p\theta_1 = p\theta_2$ and Eq. (130) reduces to

$$\mathbf{e}' = \mathbf{R}'\cdot\mathbf{i}' + \mathbf{L}'p\cdot\mathbf{i}' + p\theta\,\mathbf{N}'\cdot\mathbf{i}' \tag{132}$$

where $\mathbf{N}' = \mathbf{G}' + \mathbf{V}'$ and the subscript of θ has been deleted. Evidently, the transformation formula of **N** is the same as that of **V**.

$$\mathbf{N}' = \mathbf{C}_t\cdot\mathbf{N}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta}.$$

If $p\theta$ = a constant, then Eq. (132) can be written

$$\mathbf{e}' = \mathbf{Z}'\cdot\mathbf{i}' \tag{133}$$

where

$$\mathbf{Z}' = (\mathbf{R}' + \mathbf{L}'p + p\theta\,\mathbf{N}').$$

It can be proved that $\mathbf{N}' = \dfrac{\partial\mathbf{L}'}{\partial\theta}\cdot$ Since $\dfrac{\partial\mathbf{L}'}{\partial\theta}\,p\theta = p\mathbf{L}'$, Eq. (132) for the second primitive machine reduces to

$$\mathbf{e}' = \mathbf{R}\cdot\mathbf{i}' + p(\mathbf{L}'\cdot\mathbf{i}'). \tag{134}$$

2·69. The Transformation Formula of Z′ for Machines with Rotating Axes. To find the transformation formula for **Z** we have

$$\begin{aligned}
\mathbf{Z}' &= \mathbf{R}' + \mathbf{L}'p + p\theta_2\,\mathbf{G}' + p\theta_1\,\mathbf{V}' \\
&= (\mathbf{C}_t\cdot\mathbf{R}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\mathbf{C}p + p\theta_2\,\mathbf{C}_t\cdot\mathbf{G}\cdot\mathbf{C}) \\
&\quad + p\theta_1\left(\mathbf{C}_t\cdot\mathbf{V}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}\right) \\
&= \mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}\,p\theta_1 \text{ since } \mathbf{V} = 0 \text{ for stationary axes.}
\end{aligned}$$

Since the transformation formula for **Z** is

$$\mathbf{Z}' = \left(\mathbf{C}_t\cdot\mathbf{Z}\cdot\mathbf{C} + \mathbf{C}_t\cdot\mathbf{L}\cdot\frac{\partial\mathbf{C}}{\partial\theta_1}\,p\theta_1\right) \qquad [135]$$

evidently **Z**, for machines with rotating axes, is not a tensor.

The torque tensor for machines with rotating reference axes is

$$\mathbf{G}' = \mathbf{C}_t\cdot\mathbf{G}\cdot\mathbf{C} \qquad [136]$$

and the torque is

$$f' = \mathbf{i}'\cdot\mathbf{G}'\cdot\mathbf{i}'. \qquad [137]$$

PROBLEM XVI

1. Derive the voltage equation (Eq. 83) of the first primitive machine from Maxwell's equation

$$\mathbf{e} = \mathbf{R}\cdot\mathbf{i} + p(\mathbf{L}\cdot\mathbf{i})$$

where **L** is given by Eq. (126).

Hint: Take the **C** transformation to be the inverse of the **C** given by Eq. (138). Replace **i** in Maxwell's equation by **C**·**i**′ and carry out operations somewhat similar to those of Eq. (127–134).

(9)

Derived Machines with Rotating Reference Axes

The equations of performance of derived machines are obtained in much the same manner as explained in Sec. 7.

2·70. The C Matrix for Rotating Axes. If slip-rings exist on the machine instead of brushes, the **C** matrix is the same as in §2·62, *with the important difference that the constant angle must be replaced by the variable angle θ*, where θ defines the position of the rotor at time t. The steps in §2·62 apply in the order enumerated.

2·71. Representative Example: Two-phase Synchronous Machine. The winding diagram and the **C** matrix for the two-phase synchronous machine are respectively

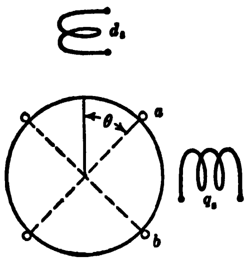

FIG. 2·30. Two-phase Synchronous Machine.

$$\begin{array}{c} \\ \mathbf{d}_s \\ \mathbf{d}_r \\ \mathbf{q}_r \\ \mathbf{q}_s \end{array} \begin{array}{cccc} \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \\ \hline 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{array} \qquad [138]$$

The metric tensor **L** and the transient impedance matrix both of the first primitive machine are given by Eq. (93) and (105) respectively. (The subscripts in Eq. 93 may be omitted.) The transient impedance matrix for the two-phase synchronous machine with moving axes is computed by Eq. (135). (Subscripts on θ will now be omitted.) The **C** matrix for the computations is given by Eq. (138). The computations yield

$$\mathbf{Z}' = \begin{array}{c|cccc} & \mathbf{d}_s & \mathbf{a} & \mathbf{b} & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds} + L_{ds}p & M_d(\cos\theta p - \sin\theta p\theta) & -M_d(\sin\theta p + \cos\theta p\theta) & 0 \\ \mathbf{a} & M_d(\cos\theta p - \sin\theta p\theta) & [r_r + (L_{dr}\cos^2\theta + L_{qr}\sin^2\theta)p + 2(L_{qr} - L_{dr})\sin\theta\cos\theta p\theta] & (L_{qr} - L_{dr})[\sin\theta\cos\theta p + (\cos^2\theta - \sin^2\theta)p\theta] & -M_q(\sin\theta p + \cos\theta p\theta) \\ \mathbf{b} & -M_d(\sin\theta p + \cos\theta p\theta) & (L_{qr} - L_{dr})[\sin\theta\cos\theta p + (\cos^2\theta - \sin^2\theta)p\theta] & [r_r + (L_{dr}\sin^2\theta + L_{qr}\cos^2\theta)p + 2(L_{dr} - L_{qr})\sin\theta\cos\theta p\theta] & M_q(\cos\theta p - \sin\theta p\theta) \\ \mathbf{q}_s & 0 & M_q(\sin\theta p + \cos\theta p\theta) & M_q(\cos\theta p - \sin\theta p\theta) & r_{qs} + L_{qs}p \end{array}$$

[139]

where p refers only to i. (See Ex. 1, problem set XVII for simplifications.)

The voltage vector $\mathbf{e}'$ is given by the first of Eqs. (129).

The voltage equation is given by Eq. (133).

The torque tensor $\mathbf{G}'$ is found by $\mathbf{G}' = \mathbf{C}_t \cdot \mathbf{G} \cdot \mathbf{C}$ or by selecting all terms in $\mathbf{Z}'$ containing $p\theta$.

The instantaneous torque f is given by

$$f = \mathbf{i}' \cdot \mathbf{G}' \cdot \mathbf{i}'. \qquad [140]$$

2·72. Transient Current Solution. In general, the voltage equations are linear differential equations with periodic coefficients. Such equations cannot be solved directly by the operational methods of Heaviside. Two cases obtain relative to the transient solution.

(*a*) *Rotating axes unessential.* If in the derivation of the equation of voltage stationary axes could have been used instead of rotating axes, then by changes of variables or by Heaviside shifting formulas the differential equations can be reduced to forms to which Heaviside's methods are applicable. In this case it is preferable to derive *ab initio* the equations of performance employing stationary axes.

(*b*) *Rotating axes essential.* If rotating axes are necessary in the derivation of the voltage equations, again two cases obtain relative to the transient solution.

(1) A sufficiently accurate transient solution may be obtained by simplifying assumptions based on physical principles. One specific tool relative to such simplifying assumptions is the constant linkage theorem.[21] An application of this theorem has been made to the set of differential equations defining (subject to certain assumptions) the transient currents of a single-phase short circuit of a synchronous generator with moving reference axes.[22]

(2) For an accurate mathematical solution recourse may be had to the advanced methods for solving analytic differential equations in Chap. III.

2·73. Steady-state Current Solution. With the exception of item (*b*) (1), the statements of §2.72 are true relative to the steady-state current solution.

For the steady-state current solution item (*b*) (1) should be replaced by the statement that it is possible under many practical conditions to

[21] R. E. Doherty, "A Simplified Method of Analyzing Short-Circuit Problems," *Trans. A.I.E.E.*, **42**, 849 (1923).

[22] R. E. Doherty and C. A. Nickle, "Synchronous Machines IV; Single-Phase Phenomena in Three-Phase Machines," *Trans. A.I.E.E.*, **47**, 457–492.

derive the steady-state impedance matrix from the transient impedance matrix. For this derivation the reader is referred elsewhere.[23]

PROBLEMS XVII

1. In the impedance matrix of Eq. (139) make the following obvious replacements:

$$M_d \cos\theta\, pi \text{ by } M_d(p\cos\theta\, i + \sin\theta\, p\theta i),$$

$$M_d(\cos\theta\, pi - \sin\theta\, p\theta i) \text{ by } M_d p\cos\theta\, i$$

and reduce $\mathbf{Z}'$ to the simpler form

	$\mathbf{d}_s$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{q}_s$
$\mathbf{d}_s$	$r_{ds} + L_{ds}p$	$pM_d \cos\theta$	$-pM_d \sin\theta$	0
$\mathbf{a}$	$pM_d \cos\theta$	$r_r + p(L_S + L_D \cos 2\theta)$	$-pL_D \sin 2\theta$	$pM_q \sin\theta$
$\mathbf{b}$	$-pM_d \sin\theta$	$-pL_D \sin 2\theta$	$r_r + p(L_S - L_D \cos 2\theta)$	$pM_q \cos\theta$
$\mathbf{q}_s$	0	$pM_q \sin\theta$	$pM_q \cos\theta$	$r_{qs} + L_{qs}p$

where $L_S = (L_{dr} + L_{qr})/2$, $L_D = (L_{dr} - L_{qr})/2$, and p refers to both $\cos\theta$ and i.

2. Write the torque tensor for the two-phase synchronous machine.

3. The winding diagram of the three-phase synchronous machine is shown in Figs. 2·31 and 2·32.

The $\mathbf{C}$ matrix is

	$\mathbf{d}_s$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{c}$	$\mathbf{q}_s$
$\mathbf{d}_s$	1	0	0	0	0
$\mathbf{d}_{r1}$	0	$\cos\theta$	0	0	0
$\mathbf{d}_{r2}$	0	0	$\cos(\theta + 120°)$	0	0
$\mathbf{d}_{r3}$	0	0	0	$\cos(\theta - 120°)$	0
$\mathbf{q}_{r3}$	0	0	0	$\sin(\theta - 120°)$	0
$\mathbf{q}_{r2}$	0	0	$\sin(\theta + 120°)$	0	0
$\mathbf{q}_{r1}$	0	$\sin\theta$	0	0	0
$\mathbf{q}_s$	0	0	0	0	1

[23] G. Kron, *The Application of Tensors to the Analysis of Rotating Electrical Machinery*, pp. 73–74. General Electric Review, 1938.

The inductance tensor **L** is given by Eq. (93) where **L** is enlarged to eight rows and eight columns. The elements of **L** are constants.

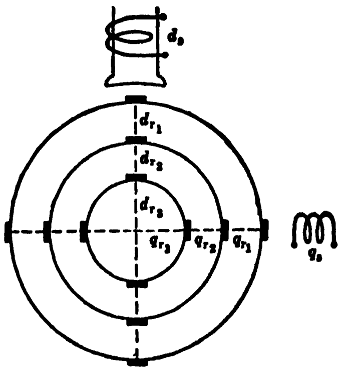

FIG. 2·31. Generalized Machine.

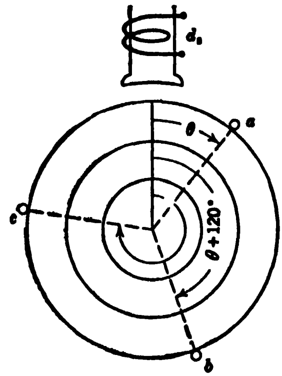

FIG. 2·32. Three-phase Synchronous Machine.

4. Compute **L′** by means of the third equation of Eqs. (129).
5. In **L′** of Ex. 4 make the substitutions

$$L_{dr} \cos^2 \theta + L_{qr} \sin^2 \theta = A + B \cos 2\theta$$

$$M_{dr} \cos \theta \cos(\theta + 120°) + M_{qr} \sin \theta \sin(\theta + 120°) = -\frac{1}{2}\frac{M_{dr} + M_{qr}}{2} + \frac{M_{dr} - M_{qr}}{2} \cos(2\theta - 120°)$$

and replace

$$L_r \text{ by } \tfrac{2}{3}(L_r + L_0/2)$$

$$M_r \text{ by } \tfrac{2}{3}(L_r - L_0)$$

$$M \text{ by } \tfrac{2}{3}M.$$

Check the linkages given by $\mathbf{L}' \cdot \mathbf{i}'$ with those given by Park.[24]

6. Derive, by the method of Sec. 9, the voltage equations of single-phase short circuits and compare with the Doherty-Nickle equations.[25]

7. Derive, by the method of Sec. 9, the voltage equations of three-phase short circuits and compare with the Doherty-Nickle equations.[26]

[24] R. H. Park, "Definition of an Ideal Synchronous Machine and Formula for the Armature Flux Linkages," *General Electric Review*, **31** (1928); "Two-Reaction Theory of Synchronous Machines," Part I, Generalized Method of Analysis, *Trans. A.I.E.E.*, **42** (1929).

[25] R. E. Doherty and C. A. Nickle, *op. cit.*, §2·72.

[26] R. E. Doherty and C. A. Nickle, "Three-Phase Short Circuit," *Trans. A.I.E.E.*, **49** (1930).

(10)

Machines Under Acceleration

In Sec. 5–9 inclusive it has been assumed that the rotor runs at constant speed. In Sec. 10 accelerated motion of the rotor is taken into account.

2·74. Equations of Voltage and Torque. When electrical machines run at a constant speed $p\theta_2 = v_2$, two invariant equations are used in their analysis; namely, the equation of voltage

$$e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + p\theta_2 G_{mn}i^n + p\theta_1 V_{mn}i^n \qquad [141]$$

(where $p\theta_1$ is the speed of the reference frame, if rotating) and the equation of torque (impressed)

$$f = -G_{mn}\, i^m\, i^n. \qquad [142]$$

Each of these equations has been established separately.

When machines have an accelerated motion (during starting or during small oscillations, etc.), the friction R and moment of inertia L also play a part in the analysis and the above equation of torque becomes (for a single machine)

$$f = Rv + L\frac{d\,p\theta}{dt} - G_{mn}\, i^m\, i^n. \qquad [143]$$

In order to study accelerated motions more conveniently, it is necessary to replace the two invariant Eqs. (141) and (142) by a single invariant equation, the so-called equation of motion, which splits up conveniently into its component equations of voltage and torque. The establishment of a single invariant equation also facilitates the analysis of rotating machines with complex structure, also the analysis of any number of interconnected machines with any type of actual or hypothetical reference frame.

2·75. The Equation of Motion. In order to establish the equation of motion, new types of geometric objects will have to be introduced whose components contain both electrical and mechanical quantities. (In Eqs. 141 and 142 each tensor contains either electrical or mechanical quantities.) For instance, for the primitive machine, the quan-

tities that are not due to motion are arranged in the following four tensors

$$e_\alpha = \begin{array}{c|c|c|c|c|}
d_s & d_r & q_r & q_s & s \\ \hline
e_{ds} & e_{dr} & e_{qr} & e_{qs} & f \\ \hline
\end{array}
\qquad
i^\alpha = \begin{array}{c|c|c|c|c|}
d_s & d_r & q_r & q_s & s \\ \hline
i^{ds} & i^{dr} & i^{qr} & i^{qs} & p\theta \\ \hline
\end{array}$$

$$R_{\alpha\beta} = \begin{array}{c|c|c|c|c|c|}
& d_s & d_r & q_r & q_s & s \\ \hline
d_s & r_{ds} & & & & \\ \hline
d_r & & r_r & & & \\ \hline
q_r & & & r_r & & \\ \hline
q_s & & & & r_{qs} & \\ \hline
s & & & & & R \\ \hline
\end{array}
\qquad
a_{\alpha\beta} = \begin{array}{c|c|c|c|c|c|}
& d_s & d_r & q_r & q_s & s \\ \hline
d_s & L_{ds} & M_d & & & \\ \hline
d_r & M_d & L_{dr} & & & \\ \hline
q_r & & & L_{qr} & M_q & \\ \hline
q_s & & & M_q & L_{qs} & \\ \hline
s & & & & & L \\ \hline
\end{array}
\qquad [144]$$

The tensor $a_{\alpha\beta}$ is called the **metric tensor.**

In general there are as many geometrical axes s as there are mechanical degrees of freedom in the system. Since the shafts of the various machines may also be interconnected by couplings, the transformation tensor $C^\alpha_{\alpha'}$, also contains geometrical axes. For the repulsion motor (Fig. 2·33)

$$C^\alpha_{\alpha'} = \begin{array}{c|c|c|c|}
& d_s & a & s \\ \hline
d_s & 1 & & \\ \hline
d_r & & \cos\alpha & \\ \hline
q_r & & \cos\alpha & \\ \hline
s & & & 1 \\ \hline
\end{array}
\qquad [145]$$

The quantities which are due to the existence of motion, namely, the torque tensor G_{mn} (that occurs twice in the equations, once giving generated voltages, the second time torques) and V_{mn} are arranged into a geometric object of valence three, $\Gamma_{\alpha\beta,\gamma}$ called the **affine connection** as shown in Fig. 2·34.

In terms of the five geometric objects e_α, i^α, $R_{\alpha\beta}$, $a_{\alpha\beta}$, and $\Gamma_{\alpha\beta,\gamma}$ the equation of motion of all electrical machines (and in general all electromechanical or electrical systems) is

$$e_\alpha = R_{\alpha\beta}\, i^\beta + a_{\alpha\beta}\frac{di^\beta}{dt} + \Gamma_{\beta\gamma,\alpha}\, i^\beta\, i^\gamma. \qquad [146]$$

The equation of power is

$$P = e_\alpha\, i^\alpha = R_{\alpha\beta}\, i^\alpha\, i^\beta + a_{\alpha\beta}\frac{di^\beta}{dt} i^\alpha + \Gamma_{\beta\gamma,\alpha}\, i^\beta\, i^\gamma\, i^\alpha. \qquad [147]$$

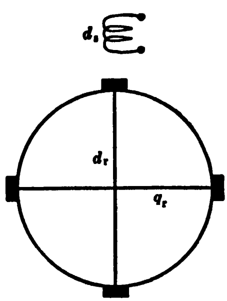

FIG. 2·33. Repulsion Motor.

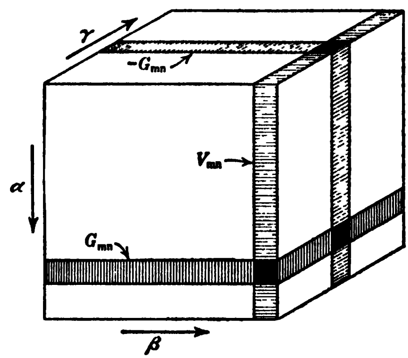

FIG. 2·34. Affine Connection, $\Gamma_{\alpha\beta,\gamma}$.

2·76. The Metric Tensor $a_{\alpha\beta}$. The equation of motion introduces three geometric objects $\Gamma_{\alpha\beta,\gamma}$, $a_{\alpha\beta}$, and $R_{\alpha\beta}$ (in addition to the vectors e_α, i^α and the scalar t) which play a basic part in the study of dynamics and geometry. The metric tensor $a_{\alpha\beta}$ plays a part in the definition of the magnitude of a vector, while the affine connection and resistance tensor $R_{\alpha\beta}$ play a part in the definition of its direction. (In the invariant equations of stationary mesh networks $e_\alpha = z_{\alpha\beta} i^\beta$ the vectors i^α, and e_α have neither magnitude nor direction. They have only components, that is, an existence.)

One of the most important concepts is the metric tensor $a_{\alpha\beta}$ representing the self- and mutual inductances and moments of inertia. When a vector A^α is given, its magnitude is defined as

$$|A|^2 = a_{\alpha\beta}\, A^\alpha\, A^\beta. \qquad [148]$$

If the vector is the generalized current vector i^α, then its magnitude is equal to $\sqrt{2T}$, where T is the total kinetic energy stored instantaneously in the system.

With the aid of the metric tensor $a_{\alpha\beta}$ it is possible to raise or lower the indices of tensors. If the inverse of $a_{\alpha\beta}$ is $a^{\alpha\beta}$, then

$$R_{\alpha\beta}\, a^{\beta\gamma} = R_{\alpha}^{\cdot\gamma} \quad \text{or} \quad R^{\alpha}_{\cdot\beta}\, a_{\alpha\gamma} = R_{\beta\gamma}. \qquad [149]$$

The indices of $\Gamma_{\alpha\beta,\gamma}$ (not being a tensor) cannot be moved. An exception is the last index, so that

$$\Gamma_{\alpha\beta,\gamma}\, a^{\gamma\delta} = \Gamma^{\delta}_{\alpha\beta}. \qquad [150]$$

The flux-linkage vector φ_α is also the covariant form of i^α and vice versa, since

$$i^\alpha\, a_{\alpha\beta} = \varphi_\beta = i_\beta \quad \text{and} \quad \varphi_\alpha\, a^{\alpha\beta} = \varphi^\beta = i^\beta. \qquad [151]$$

Also

$$2T = a_{\alpha\beta}\, i^\alpha\, i^\beta = \varphi_\alpha\, i^\alpha = i_\alpha\, i^\alpha = \varphi_\alpha\, \varphi^\alpha = |\,\varphi\,|^2 = |\,i\,|^2.$$

That is, the current vector i^α and the flux-linkage vector φ_α are the contravariant and the covariant representation ("extensity" and "intensity" factors) of the same physical entity, the "stored kinetic energy" T.

Tensors having the same base letters but having indices in different positions, as $R_{\alpha\beta}$ or $R^{\alpha}_{\cdot\beta}$ or $R_{\alpha}^{\cdot\beta}$ or $R^{\alpha\beta}$ are called **associated tensors.** The components of $R^{\alpha}_{\cdot\beta}$, however, do not represent resistances but "decrement factors" δ, and the components of $\Gamma^{\alpha}_{\beta\gamma}$ do not represent self- and mutual inductances but "leakage coefficients" where

$$\delta = \frac{r}{L} = \frac{\text{resistance}}{\text{inductance}} \quad \text{and} \quad \lambda = \frac{\text{self-inductance}}{\text{mutual inductance}}. \qquad [152]$$

The use of *ratios* (generalized per unit quantities) in place of actual design constants facilitates the comparison of machines of different sizes, supplies a ready-made method to find the locus-diagrams graphically, and in general simplifies the algebraic and numerical calculations

In terms of associated tensors, another form of the equations of motion is

$$e^\alpha = R^{\alpha}_{\cdot\beta}\, i^\beta + \frac{di^\beta}{dt} + \Gamma^{\alpha}_{\beta\gamma}\, i^\beta\, i^\gamma, \qquad [153]$$

where

$$i^\alpha = \frac{dq^\alpha}{dt}.$$

2·77. The Component Parts of the Affine Connection $\Gamma_{\alpha\beta,\gamma}$. The affine connection $\Gamma_{\alpha\beta,\gamma}$ appears in the equation of electrical machines because of the existence of mechanical motion and it contains all the additional self- and mutual inductances that appear between the ter-

minals by the presence of these motions. (In general these inductances are independent of those of the components of $a_{\alpha\beta}$, that appear because of the motion of electric charges.)

In general there are at least three different types of motion in a system of rotating electrical machines (besides the motion of the electrical charges): (1) Conductors rotate. (2) The magnetic paths rotate. (3) The reference frames (real or hypothetical) rotate.

Each of these motions introduces a different set of self- and mutual inductances that are arranged in each reference frame into a cube, forming part of $\Gamma_{\alpha\beta,\gamma}$. Each of these component parts forms a separate geometric object, so that $\Gamma_{\alpha\beta,\gamma}$ is itself a sum of three different types of geometric objects. In particular:

(*a*) The inductances due to rotation of the *conductors* may be arranged into the "torsion tensor" $S_{\alpha\beta\gamma}$. It is a tensor of valence three skew-symmetric in its first two indices

$$S_{\alpha\beta\gamma} = -S_{\beta\alpha\gamma} \qquad [154]$$

since it contains only G_{mn} and $-G_{mn}$.

In machines in which the flux-density waves in the rotor are not sinusoidal in space (such as in direct-current and alternating-current commutator machines), the components of $S_{\alpha\beta\gamma}$ are independent of the components of $a_{\alpha\beta}$ and there is no relation between $S_{\alpha\beta\gamma}$ and $a_{\alpha\beta}$. However, in machines with sinusoidal rotor-flux densities (such as synchronous and induction machines), the components of $S_{\alpha\beta\gamma}$ may be found (for the primitive machine only) from those of a by the formula

$$S_{\alpha\beta\gamma} = \frac{1}{2} a_{\gamma\delta}\, C_{\alpha}^{\alpha'}\, C_{\beta}^{\beta'} \left(\frac{\partial C_{\beta'}^{\delta}}{\partial x^{\alpha'}} - \frac{\partial C_{\alpha'}^{\delta}}{\partial x^{\beta'}}\right), \qquad [155]$$

where $C_{\alpha'}^{\alpha}$ changes the slip-ring axes to direct and quadrature axes and $C_{\alpha'}^{\alpha}$ is a function only of the displacement x^{α} of the rotor or rotors.

(*b*) The inductances due to the rotation of the *magnetic paths* (salient poles) may be arranged into a cube $\partial a_{\alpha\beta}/\partial x^{\gamma}$. This quantity cannot be denoted by one symbol since it is not a geometric object in general.

When the flux due to the rotating magnetic paths is non-sinusoidal in space, the inductances due to the additional non-sinusoidal portions may be arranged into a tensor of rank three, $Q_{\alpha\beta\gamma}$. This tensor has no special name. It is symmetrical in its last two indices.

$$Q_{\alpha\beta\gamma} = Q_{\alpha\gamma\beta}. \qquad [156]$$

Hence the magnetic paths contribute $\partial a_{\alpha\beta}/\partial x^{\gamma}$ and $Q_{\alpha\beta\gamma}$.

(*c*) The inductances due to the rotation of the *reference frames* are arranged into a geometric object, the non-holonomic object. It is defined by a formula analogous to Eq. (155), namely,

$$\Omega_{\alpha\beta,\gamma} = \frac{1}{2} a_{\gamma\delta} C_\alpha^{\alpha'} C_\beta^{\beta'} \left(\frac{\partial C_{\alpha'}^\delta}{\partial x^{\beta'}} - \frac{\partial C_{\beta'}^\delta}{\partial x^{\alpha'}}\right), \qquad [157]$$

where now the components of $C_{\alpha'}^\alpha$ are functions of the displacements x^α that differ from those of the rotors.

2·78. Definition of the Affine Connection $\Gamma_{\alpha\beta,\gamma}$. The affine connection is built up from four sets of inductances:

(*a*) The motion of the conductors introduces $S_{\alpha\beta\gamma}$.

(*b*) The motion of the magnetic paths introduces $\partial a_{\alpha\beta}/\partial x^\gamma$ and $Q_{\alpha\beta\gamma}$.

(*c*) The motion of the reference frames introduces $\Omega_{\alpha\beta,\gamma}$.

In the definition of $\Gamma_{\alpha\beta,\gamma}$ each of the above four quantities occurs three times, with their indices arranged in the same even permutation $\alpha\beta\gamma$, $\beta\gamma\alpha$, and $\gamma\alpha\beta$. That is,

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right) + S_{\alpha\beta\gamma} - S_{\beta\gamma\alpha} + S_{\gamma\alpha\beta}$$
$$+ \tfrac{1}{2}\left(Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} - Q_{\gamma\alpha\beta}\right) - \Omega_{\alpha\beta,\gamma} + \Omega_{\beta\gamma,\alpha} - \Omega_{\gamma\alpha,\beta}. \qquad [158]$$

This is the most general form of $\Gamma_{\alpha\beta,\gamma}$ that is used in tensor analysis (in affine differential geometry). Its formula of transformation is

$$\Gamma_{\alpha'\beta',\gamma'} = \Gamma_{\alpha\beta,\gamma} C_{\alpha'}^\alpha C_{\beta'}^\beta C_{\gamma'}^\gamma + a_{\gamma\alpha} C_{\gamma'}^\gamma \frac{\partial C_{\alpha'}^\alpha}{\partial x^{\beta'}}. \qquad [159]$$

The expression in parenthesis is called the **Christoffel symbol.** It is also a geometric object of valence three

$$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right). \qquad [160]$$

Its transformation formula is the same as that of $\Gamma_{\alpha\beta,\gamma}$ of Eq. (159). It is customary to include the non-holonomic object $\Omega_{\alpha\beta,\gamma}$ in the definition of $[\alpha\beta,\gamma]$ and call it the **non-holonomic form of the Christoffel symbol** as

$$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right) - \Omega_{\alpha\beta,\gamma} + \Omega_{\gamma\beta,\alpha} - \Omega_{\gamma\alpha,\beta}. \qquad [160a]$$

Its law of transformation is still the same as that of $\Gamma_{\alpha\beta,\gamma}$.

In special cases $\Gamma_{\alpha\beta,\gamma}$ assumes simpler forms. In all problems of classical mechanics $S_{\alpha\beta\gamma}$ and $Q_{\alpha\beta\gamma}$ are zero and in most problems $\Gamma_{\alpha\beta,\gamma}$

is also zero, so that $\Omega_{\alpha\beta,\gamma}$ is identical with the holonomic Christoffel symbol $[\alpha\beta,\gamma]$ Eq. (160). In most standard electrical machines with stationary axes $[\alpha\beta,\gamma]$, $Q_{\alpha\beta\gamma}$ and $\Omega_{\alpha\beta,\gamma}$ are zero, but not $S_{\alpha\beta\gamma}$.

2·79. Permanent Magnets. When permanent magnets are present in the system (as in the case of the numerous types of subsynchronous motors), they introduce an additional flux-linkage vector φ_α that is not a function of the currents i^α. Their presence introduces a skew-symmetric tensor of valence two:

$$M_{\alpha\beta} = \frac{\partial\varphi_\alpha}{\partial x^\beta} - \frac{\partial\varphi_\beta}{\partial x^\alpha}, \qquad [161]$$

which may be combined with the symmetrical resistance tensor $R_{\alpha\beta}$ to form a general tensor of valence two:

$$B_{\alpha\beta} = R_{\alpha\beta} + M_{\alpha\beta}. \qquad [162]$$

In any reference frame the symmetrical part of $B_{\alpha\beta}$ gives $R_{\alpha\beta}$ and its skew-symmetric part gives $M_{\alpha\beta}$.

2·80. The Most General Form of the Equation of Motion. Although in the definition of $\Gamma_{\alpha\beta,\gamma}$ each of the four quantities $\partial a_{\alpha\beta}/\partial x^\gamma$, $S_{\alpha\beta\gamma}$, $Q_{\alpha\beta\gamma}$ and $\Omega_{\alpha\beta,\gamma}$ occurs three times, in the equation of motion $\Gamma_{\alpha\beta,\gamma}$ appears multiplied by i^α twice as $\Gamma_{\alpha\beta,\gamma}\, i^\alpha\, i^\beta$ and consequently some of the quantities disappear or simplify. That is, the equation of motion of rotating electrical machinery may be written as

$$e_\gamma = B_{\gamma\beta}\, i^\beta + a_{\gamma\beta}\frac{di^\beta}{dt} + ([\alpha\beta,\gamma] - 2S_{\beta\gamma\alpha} + Q_{\alpha\beta\gamma} - \tfrac{1}{2}\, Q_{\alpha\beta\gamma})\, i^\alpha\, i^\beta, \qquad [163]$$

where $i^\alpha = dq^\alpha/dt$ and $[\alpha\beta,\gamma]$ is the non-holonomic Christoffel symbol of Eq. (160a) simplified to

$$[\alpha\beta,\gamma] = \frac{\partial a_{\beta\gamma}}{\partial x^\alpha} - \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial x^\gamma} + 2\,\Omega_{\gamma\beta,\alpha} \qquad [160b]$$

This is the most general form of the equation of motion that is used in affine differential geometry where it represents (when t is replaced by the arc length s and $e_\gamma = 0$) the equation of the shortest lines between two points (paths) in a curved affine space.

It is emphasized that the physical interpretation given above (namely, that each term in the equation represents the voltages and torques due to the motion of some particular medium) is valid only if the first primitive machine is used as the primary reference machine to find the equations of some particular machine. However, if some other machine, say one whose reference axes are connected to the moving conductors, is assumed as the primary reference machine (which is,

of course, allowable), then the above equation is still valid, but the physical interpretation of each term is far more complicated. Each term then represents the voltages and torques due to several of the moving materials instead of one.

Sections 1–10 of this chapter are but an introduction to the tensor theory of networks and rotating electrical machinery. The literature of the field is extensive. A number of references are given in Sec. 12.

(11)

Tensorial Method of Attack of Engineering Problems

The question arises as to how the engineer can put tensor equations to practical use. This question has already been answered by engineers having put them to use. However, it may be helpful to summarize briefly the process.

2·81. Derivation. Only the derivation of equations of performance is here considered. Suppose the engineer is called upon to analyze a complicated engineering structure such as the hunting of a turbine governing system or an electric drive. The steps in the tensorial method of establishing the equations are as follows.

(*A*)

1. Do not attempt to analyze immediately the given system, since it is too complex.

2. Instead, subdivide the complex system into smaller component parts, the primitive system (for a simple example, see §2·42) where (*a*) the equations of each component have already been established before (see §§2·39, 2·40), or if not previously analyzed, then (*b*) it is comparatively easy to establish the equations of each part either by further subdivision or by assuming more convenient reference frames, or by any other means. The subdivision may be accomplished in one or more steps depending upon the complexity of the resultant and the component structures. In addition to subdividing the system, new and more easily analyzable reference frames can be assumed.

There is no necessity to assume the existence of reaction forces acting at the points or planes where the original structure was broken up. That is, each component system is analyzed as if the other component systems are non-existent. (See §§2·39, 2·40.)

3. Establish the tensor equations of the primitive system consisting of several isolated structures. (See Eqs. 70, 71, 72.)

(*B*)

Set up the connection tensor (transformation tensor) showing how the component parts are interconnected into the actual system and also how the actual reference frames differ from the simplified ones. (Eqs. 74.)

(*C*)

Transform each tensor of the primitive system with the aid of the connection tensor. (Eq. 75, for example.) Since the tensor equation of the original complex system is the same as that of the simpler primitive system, the equations of the given engineering structure have been established.

The material of §§2·39–2·42 illustrates, in a simple case, the procedure described. For a complex illustration the reader is referred to Ref. 2 of §2·82.

(12)

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CHAPTER III

NON-LINEARITY IN ENGINEERING

(1) Differential Equations Analytic in Parameters, (2) Non-linear Systems by Variations of Parameters, (3) Solutions of Systems by Method of Successive Integrations, (4) Matrix Methods, (5) Elliptic Functions, (6) Hyperelliptic Functions, (7) Method of Collocation, (8) Galerkin's Method, (9) Method of Lalesco's Non-linear Integral Equations, (10) Solutions by the Differential Analyzer, (11) Additional Methods and References.

The first two chapters of this text were concerned with the analytical development of certain fundamental principles of mathematical engineering and the reduction of engineering problems to mathematical systems by means of these fundamental principles. Solutions of the resulting discrete systems may or may not depend upon advanced mathematics. If the solutions required no mathematics beyond the domain of elementary differential equations, Heaviside's operational calculus, or the elementary theory of matrices the solutions were completed in Chaps. I and II.

Engineering problems of considerable difficulty may lead to mathematically discrete systems whose solutions depend upon advanced mathematics. Such problems frequently reduce to systems of non-linear differential or non-linear integral equations.

In general, a non-linear problem is one which, when formulated mathematically, reduces to (one or) a system of differential, integral, or integro-differential equations such that at least one of the three quantitives, a derivative, an integral, or a dependent variable, is involved transcendentally or in some manner to a power higher than the first in at least one equation of the system. From Part I it is evident that analyses of investigations in circuits, electrical machines, heat-flow, elasticity, and dynamical systems lead more and more to systems of differential and integral equations whose dependent variables and (or) their derivatives are involved to a power higher than the first. The differential equations present such a variety of types that the so-called standard forms of differential equations studied in a first course in dif-

ferential equations are of slight use for the simple reason that they fail to arise in difficult problems in engineering practice. Engineering non-linear problems are most often reducible mathematically to the solution of systems of non-linear differential equations and non-linear integral equations. It is the purpose of this chapter to explain briefly the theory of these systems and, what is more important from an engineering point of view, to apply them in the solution of practical problems in engineering.

(1)

Differential Equations Analytic in Parameters

The general theory of differential equations analytic in parameters is, in general, conveniently applicable to equations in the so-called normal form.

3·1. Reduction of Systems of Differential Equations to Normal Form. The normal form consists of a system of simultaneous differential equations, the left members containing a single first derivative, while the right members contain no derivative. The number of equations in the normal form of the system equals the order of the system, i.e., the number of constants in the solution. Reduction to the normal form is merely a routine process. One new dependent variable must be introduced for each derivative of order higher than the first which occurs. The process is illustrated in the example following.

EXAMPLE. The differential equations of motion of a projectile, under proper conditions, are

$$\frac{d^2x}{dt^2} = -k^2\frac{dx}{dt},$$

$$\frac{d^2y}{dt^2} = -k^2\frac{dy}{dt} - g,$$

where x, y are the coordinates of the projectile, t is the time, and

$$k^2 = \frac{H(y)G(v)}{C}.$$

The constant C is the ballistic coefficient dependent upon the shape of the projectile, $H(y)$ is a function of the height of the shell above ground and $G(v)$ is a function of the velocity.

Reduce these equations to the normal form. Let

$$x = x_1, \quad y = x_3,$$

$$\frac{dx}{dt} = x_2, \quad \frac{dy}{dt} = x_4.$$

Then

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_3}{dt} = x_4,$$

$$\frac{dx_2}{dt} = -k^2x_2, \quad \frac{dx_4}{dt} = -k^2x_4 - g.$$

The last four equations are the normal form of the two second-order differential equations of the motion of a projectile.

3·2. Equations of Type II. Let the system of differential equations, as given by the physical problem, be reduced to the normal form

$$\begin{gathered} x_i' = F_i(x_1, x_2, \cdots, x_n; t) = F_i(x_j; t) \quad (i, j = 1, 2, \cdots, n) \\ x_i(t_0) = a_i \quad \text{where} \quad x_i' = \frac{dx_i}{dt}. \end{gathered} \tag{1}$$

In Eqs. (1) the second system of n equations, namely, $x_i(t_0) = a_i$ are, of course, the n initial conditions. If (1) contain a parameter r or if a parameter r can be introduced by change of variables in such a way that (1) are reducible to the form

$$\begin{gathered} x_i' = f_i(x_j; t) + r\, g_i(x_j, r; t) \quad (i, j = 1, 2, \cdots, n) \\ x_i(t_0) = a_i \end{gathered} \tag{2}$$

then the system is said to be of type II. The properties of the functions f_i and g_i are described presently.

First consider the system of equations

$$\begin{gathered} x_i' = f_i(x_j; t), \quad (i, j = 1, 2, \cdots, n) \\ x_i(t_0) = a_i, \end{gathered} \tag{3}$$

where the functions f_i are analytic in x_j and t for all x_j and t which satisfy the relations

$$|x_j - a_j| \leqq r_j, \quad |t - t_0| \leqq T_0. \tag{4}$$

Equations (4) state merely that the n functions f_i are analytic in the interior and on the boundary of some $(n + 1)$ dimensional region. This condition is usually satisfied in engineering problems. A function $f(x_1, x_2, \cdots x_n; t)$ is **analytic** in the region specified by (4) if it is uniquely expansible in a power series in the $(n + 1)$ variables $(x_j - a_j)$

and $(t - t_0)$ and if the series is convergent in the region defined by (4). The method of obtaining this expansion is given in §3·5.

It is provable[1] that (3) possess a unique continuous solution $x_i = x_i^{(0)}(t)$. If, in (2), the $g_i(x_j, r;t)$ are analytic in $x_j - x_j^{(0)}$ and r uniformly with respect to t and are continuous in t for all x_j, r, and t in the region

$$|x_j - x_j^{(0)}| \leqq r_j, \quad r \leqq \rho, \quad t_0 \leqq t \leqq T \leqq T_0 \qquad [5]$$

then there exists a formal solution[2] of (2) of the form

$$x_i = x_i^{(0)}(t) + x_i^{(1)}(t)r + x_i^{(2)}(t)r^2 + \cdots, (i = 1, 2, \cdots, n) \qquad [6]$$

where $x_i^{(0)}$, $x_i^{(1)}$, $\cdots$ are determined in §3·5.

The notation in (6) may require explanation. The functions $f_i(x_j;t)$ are functions of $(n + 1)$ variables of which n of them are $x_1, x_2, \cdots, x_n$. Each x_j, by (6), is the sum of infinitely many explicit functions of t. Hence either an additional subscript or some other device must be employed to arrange in order the set of functions for each x_j. It is convenient and customary to use superscripts.

3·3. Nature of the Solution of Type II. Systems of type II are useful especially in solving engineering problems in which $rg_i(r_j,r;t)$ are less in absolute value then $f_i(x_j,r;t)$. Series (6) is then usually rapidly convergent and the terms linear and quadratic in r furnish sufficient accuracy. If the system contains no parameter r, one can frequently be introduced by change of dependent or independent variables. If it is evident that the solution $x_i = x_i^{(0)}(t)$, which is called the **generating solution** of (2), is not even an approximate solution of (2), then the method may yield such complicated results that they may be of little engineering value. It is then necessary to resolve F_i into f_i and g_i in a different manner or resort to methods of the sections which follow. The guide in resolving F_i into the sum of two functions is the physics of the problem.

During transient performance of rotating synchronous machines[3] the effect of field and armature resistance on fluxes is small if the time is sufficiently small. In this case the constant leakage theorem[4] may

[1] E. L. Ince, *Ordinary Differential Equations*, Chap. III.

[2] A formal solution is one which merely satisfies the differential equations when substituted therein. The solution may be a divergent series of such a nature that it does not define a function. A formal solution may be worthless.

[3] R. E. Doherty and C. A. Nickle, "Synchronous Machines IV," *Trans. A.I.E.E.*, **47** (April, 1928).

[4] R. E. Doherty, "A Simplified Method of Analyzing Short-Circuit Problems," *Trans. A.I.E.E.*, **42** (1923); "Short-Circuit Current of Induction Motors and Generators," *ibid.*, **40** (1921).

furnish the generating solution $x_i = x_i^{(0)}(t)$, i.e., the solution of (3). Steinmetz has given the rate of build-up of field flux[5] of a compound or shunt generator running at constant speed. This solution can be used as the generating solution for the more general case of a machine running at variable speed. In the differential equations of dynamic braking of a synchronous machine the most complicated term in the differential equations contributes only ten per cent of the solution. (That is, a solution computed without the complicated term gives a result which is 90 per cent accurate when an oscillogram is used as an answer book.) The equations in this case are reducible to type II.

Non-linear problems are problems of great difficulty. In problems of this type, solutions in closed form (a closed solution is a non-series analytical solution) are not to be expected. Indeed it is sometimes provable that no solutions in closed form exist. Series solutions may not possess the elegance of form that solutions of differential equations with constant coefficients possess, but if the solution contains the parameters of the problem in such a way that performance of the physical system can be predicted, then it is sufficient.

3·4. Introductory Example. Obtain the solution, as far as the terms linear in r, of the system of differential equations

$$\begin{aligned} x_1' &= -x_2, \\ x_2' &= x_1 + r\,x_2^2, \end{aligned} \tag{7}$$

with the initial conditions

$$\begin{aligned} x_1(0) &= 0, \\ x_2(0) &= -1. \end{aligned}$$

There exists a solution of (7) of the form of (6). The substitution of (6) in (7) yields

$$x_1^{(0)\prime} + x_1^{(1)\prime} r + \cdots = -(x_2^{(0)} + x_2^{(1)} r + \cdots),$$

$$x_2^{(0)\prime} + x_2^{(1)\prime} r + \cdots = x_1^{(0)} + x_1^{(1)} r + \cdots + r(x_2^{(0)} + x_2^{(1)} r + \cdots)^2$$

or equating coefficients of like powers of r

$$\begin{aligned} &\begin{cases} x_1^{(0)\prime} = -\,x_2^{(0)}, \\ x_2^{(0)\prime} = x_1^{(0)}, \end{cases} \\ &\begin{cases} x_1^{(1)\prime} = -\,x_2^{(1)} \\ x_2^{(1)\prime} = x_1^{(1)} + (x_2^{(0)})^2, \end{cases} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{8}$$

[5] C. P. Steinmetz, *Transient Phenomena*, p. 32.

The first pair of (8) is equivalent to $x_1^{(0)''} + x_1^{(0)} = 0$ whose general solution is

$$x_1^{(0)} = A_0 \sin t + B_0 \cos t.$$

From

$$x_1^{(0)'} = -x_2^{(0)}$$

there follows

$$x_2^{(0)} = -A_0 \cos t + B_0 \sin t$$

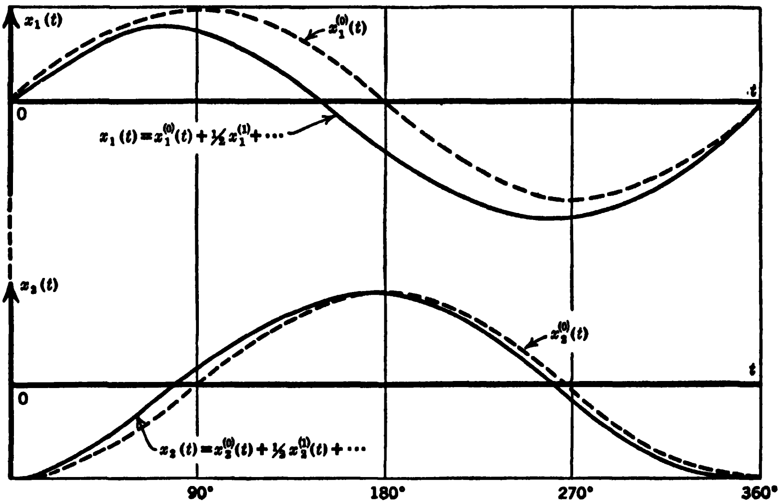

FIG. 3·1. Solution and Generating Solution.

The initial conditions, when substituted in the general solution, give

$$x_1^{(0)} = \sin t,$$
$$x_2^{(0)} = -\cos t.$$

Substituting $x_2^{(0)}$ in the second pair of Eqs. (8) and solving the resulting equations, we have

$$x_1^{(1)} = A_1 \sin t + B_1 \cos t - \tfrac{1}{2} + \tfrac{1}{6} \cos 2t,$$
$$x_2^{(1)} = -A_1 \cos t + B_1 \sin t + \tfrac{1}{3} \sin 2t.$$

Since the initial conditions of the problem have been satisfied by $x_1^{(0)}(0) = 0$, and $x_2^{(0)}(0) = -1$, it follows that $x_1^{(1)}(0) = x_2^{(1)}(0) = 0$. Applying these conditions to the general solution for $x_1^{(1)}$ and $x_2^{(1)}$, we obtain

$$x_1^{(1)} = -\tfrac{1}{2} + \tfrac{1}{3} \cos t + \tfrac{1}{6} \cos 2t,$$
$$x_2^{(1)} = \tfrac{1}{3}(\sin t + \sin 2t).$$

The required solution of (7) is

$$\begin{aligned} x_1 &= \sin t - \tfrac{1}{6}(3 - 2\cos t - \cos 2t)r + \cdots. \\ x_2 &= -\cos t + \tfrac{1}{3}(\sin t + \sin 2t)r + \cdots \end{aligned} \qquad [9]$$

The graphs of both the generating and complete solution for $r = \frac{1}{2}$ are plotted in Fig. 3·1.

EXERCISE I

1. Reduce the equation $x'' + (a + rbx)x' + cx = 0$ to the normal form. Obtain a formal solution, by the method of § 3·4, of the resulting normal system under the assumption that a and c are very large relative to r and b. The motion is oscillatory in the physical problem. (The equation is a simplified equation of hunting.) Take as initial conditions that the displacement $x_1 = 0$ at $t = 0$ and the velocity $x_2 = k$, a small quantity, at $t = 0$. Two terms of each of the two series are sufficient.

2. Obtain, by the method of § 3·4, a formal solution of the system

$$x_1' = x_2 + r\, x_1 x_2^2,$$

$$x_2' = -(1 - r)x_1 + r\, x_1^2 x_2,$$

where r is less than unity. Choose the initial conditions such that the solution is simple. Two terms of each of the series are sufficient. Find a physical system of which the above differential equations are the equations of performance.

3. Obtain a formal solution, subject to the initial conditions $i(0) = k$, $i'(0) = 0$ and for the interval $0 \leqq t \leqq m < 1$, of the differential equation

$$i'' + i + ri^3 + r^2\sqrt{1 - r^2}\, i^5 = 0,$$

where $0 < r < \frac{1}{2}$. Three terms of the series in r are sufficient. Find a mechanical system of which this differential equation is the equation of motion. Find an electrical system of which this is the equation of performance.

3·5. General Theory of Equations of Type II. The success of the method of integration in powers of a parameter depends upon the resolution of F_i of (1) into f_i and g_i such that (3) are integrable in suitable form. It is supposed then that the solution of system (3) has been obtained. This solution $x_i = x_i^{(0)}(t)$ is the generating solution of (2).

In complicated problems it is necessary to expand f_i and g_i as power series in $(x_j - x_j^{(0)})$ and r. A proof of Taylor's expansion of a function of several variables is recalled in order to emphasize the distinction between the expansion of a function in powers of $(x_j - a_j)$ where a_j are constants and in powers of $(x_j - x_j^{(0)})$ where $x_j^{(0)}$ are functions of t. A function of two independent variables and one parameter is sufficient to display the reasoning.

First let $f(x_1, x_2, r)$ be expanded in powers of $x_j - a_j$ and r. Suppose that $f(x_1, x_2, r)$ and its first n partial derivatives are continuous in the region $|x_j - a_j| \leqq h_j$, and $0 < r \leqq R$. Let

$$x_j = a_j + h_j s \quad \text{and} \quad r = \rho s \tag{10}$$

where a_j, h_j, and ρ are constants and s is a variable which lies in the interval $0 \leqq s \leqq 1$. Define a new function

$$F(s) = f(a_1 + h_1 s, a_2 + h_2 s, \rho s) = f(x_1, x_2, r).$$

The expansion of $F(s)$ in a Maclaurin series is

$$F(s) = F(0) + F'(0)s + F''\frac{(0)s^2}{2!} + F'''\frac{(0)s^3}{3!} + \cdots \tag{11}$$

The total derivatives of $F'(s)$, $F''(s)$, $\cdots$ are

$$\begin{aligned}
F'(s) &= f_{x_1}(x_1, x_2, r)\frac{dx_1}{ds} + f_{x_2}(x_1, x_2, r)\frac{dx_2}{ds} + f_r(x_1, x_2, r)\frac{dr}{ds} \\
&= h_1 f_{x_1}(x_1, x_2, r) + h_2 f_{x_2}(x_1, x_2, r) + \rho f_r(x_1, x_2, r), \\
F''(s) &= [h_1 f_{x_1x_1}(x_1, x_2, r) + h_2 f_{x_2x_1}(x_1, x_2, r) + \rho f_{rx_1}(x_1, x_2, r)]\frac{dx_1}{ds} \\
&\quad + [h_1 f_{x_1x_2}(x_1, x_2, r) + h_2 f_{x_2x_2}(x_1, x_2, r) + \rho f_{rx_2}(x_1, x_2, r)]\frac{dx_2}{ds} \\
&\quad + [h_1 f_{x_1r}(x_1, x_2, r) + h_2 f_{x_2r}(x_1, x_2, r) + \rho f_{rr}(x_1, x_2, r)]\frac{dr}{ds} \\
&= [h_1^2 f_{x_1x_1}(x_1, x_2, r) + h_2^2 f_{x_2x_2}(x_1, x_2, r) + \rho^2 f_{rr}(x_1, x_2, r) \\
&\quad + 2h_1h_2 f_{x_1}f_{x_2}(x_1, x_2, r) + 2h_1\rho f_{rx_1}(x_1, x_2, r) \\
&\quad + 2h_2\rho f_{rx_2}(x_1, x_2, r)],
\end{aligned}$$

where

$$f_{x_1} = \frac{\partial f}{\partial x_1}, \quad f_{x_1x_1} = \frac{\partial^2 f}{\partial x_1^2}, \quad f_{x_1x_2} = \frac{\partial^2 f}{\partial x_1\,\partial x_2}, \quad \text{etc.}$$

The values of $F(0)$, $F'(0)$, $F''(0)$ are

$$\begin{aligned}
F(0) &= f(a_1, a_2, 0) \\
F'(0) &= h_1 f_{x_1}(a_1, a_2, 0) + h_2 f_{x_2}(a_1, a_2, 0) + \rho f_r(a_1, a_2, 0) \\
F''(0) &= h_1^2 f_{x_1x_1}(a_1, a_2, 0) + h_2^2 f_{x_2x_2}(a_1, a_2, 0) \\
&\quad + \rho^2 f_{rr}(a_1, a_2, 0) + 2h_1h_2 f_{x_1x_2}(a_1, a_2, 0) \\
&\quad + 2h_1\rho f_{rx_1}(a_1, a_2, 0) + 2h_2\rho f_{rx_2}(a_1, a_2, 0),
\end{aligned}$$

. . . .

When these values are substituted in (11)

$$\begin{aligned}F(s) &= f(a_1, a_2, 0) + [h_1 f_{x_1}(a_1, a_2, 0) + h_2 f_{x_2}(a_1, a_2, 0)\\ &\quad + \rho f_r(a_1, a_2, 0)]s + \frac{1}{2!}[h_1^2 f_{x_1x_1}(a_1, a_2, 0)\\ &\quad + h_2^2 f_{x_2x_2}(a_1, a_2, 0) + \rho^2 f_{rr}(a_1, a_2, 0)\\ &\quad + 2h_1h_2 f_{x_1x_2}(a_1, a_2, 0) + 2h_1\rho f_{rx_1}(a_1, a_2, 0)\\ &\quad + 2h_2 f_{rx_2}\rho(a_1, a_2, 0)]s^2 + \cdots\end{aligned}$$

The last equation may be written

$$\begin{aligned}F(s) &= f(a_1, a_2, 0) + \left(sh_1\frac{\partial}{\partial x_1} + sh_2\frac{\partial}{\partial x_2} + s\rho\frac{\partial}{\partial r}\right)^1 f(x_1, x_2, r)\\ &\quad + \frac{1}{2!}\left(sh_1\frac{\partial}{\partial x_1} + sh_2\frac{\partial}{\partial x_2} + s\rho\frac{\partial}{\partial r}\right)^2 f(x_1, x_2, r) + \cdots\end{aligned}$$

where it is understood that after the operator $\left(sh_1\frac{\partial}{\partial x_1} + sh_2\frac{\partial}{\partial x_2} + s\rho\frac{\partial}{\partial r}\right)$ has been raised to the power indicated and the partial derivatives taken, then the variables x_1, x_2, r in f are replaced by a_1, a_2, and 0.

From (10) $sh_j = x_j - a_j$ and $s\rho = r$. If these substitutions are made in the value for $F(s)$, there results

$$\begin{aligned}f(x_1, x_2, r) &= f(a_1, a_2, 0) + \left[\sum_{j=1}^{2}(x_j - a_j)\frac{\partial}{\partial x_j} + r\frac{\partial}{\partial r}\right]f(x_1, x_2, r)\\ &\quad + \frac{1}{2!}\left[\sum_{j=1}^{2}(x_j - a_j)\frac{\partial}{\partial x_j} + r\frac{\partial}{\partial r}\right]^2 f(x_1, x_2, r) + \cdots \qquad [12]\end{aligned}$$

If the independent variables are $x_1, x_2, \cdots, x_n$ then the summations in (12) range from $j = 1$ to $j = n$.

The development (12) is valid in the vicinity of the point $(a_1, a_2, 0)$ and holds for all points within the parallelepiped $|x_j - a_j| \leqq h_j$, $0 < r \leqq R$. The solution $x_i = x_i^{(0)}(t)$ of (3) defines a curve in space. If in (12) a_j is replaced by $x_j^{(0)}(t)$ and the function f and its derivatives satisfy continuity conditions then the expansion becomes

$$\begin{aligned}f(x_1, x_2, r) &= f(x_1^{(0)}, x_2^{(0)}, 0) + \left[\sum_{i=1}^{2}(x_i - x_i^{(0)})\frac{\partial}{\partial x_i} + r\frac{\partial}{\partial r}\right]f(x_1, x_2, r)\\ &\quad + \frac{1}{2!}\left[\sum_{i=1}^{2}(x_i - x_i^{(0)})\frac{\partial}{\partial x_i} + r\frac{\partial}{\partial r}\right]^2 f(x_1, x_2, r) + \cdots \qquad [13]\end{aligned}$$

where (x_1, x_2, r) in f are replaced by $(x_i^{(0)}, 0)$ after the indicated operations have been carried out.

Upon expanding $f_i(x_j;t)$ and $g_i(x_j,r;t)$ of (2) in powers of $(x_j - x_j^{(0)})$ and r by (13); replacing $(x_j - x_j^{(0)})$, in these expansions, by $x_j^{(1)}(t)r + x_j^{(2)}(t)r^2 + \cdots$ from (6); substituting these results and (6) in (2) and finally equating corresponding powers of r on the two sides of the equations, it is found that

$$\frac{dx_i^{(0)}}{dt} = f_i(x_j^{(0)};t) \quad (i = 1, 2, \cdots, n)$$

$$\frac{dx_i^{(1)}}{dt} - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} x_j^{(1)} = g_i(x_j^{(0)};t), \tag{14}$$

$$\frac{dx_i^{(2)}}{dt} - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} x_j^{(2)} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_k} x_j^{(1)} x_k^{(1)} + \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_j^{(1)} + \frac{\partial g_i}{\partial r},$$

. . . .

where x_j and r in f_i and g_i are replaced by $x_j^{(0)}$ and 0 after the differentiations have been performed.

Criteria for the convergence of the series of (6) are given in §3·7.

The $x_i^{(0)}$ in (13) is a function of t in the interval $t_0 \leqq t \leqq t_1$. If the series (13) converges for all values of r and for every value of t in $t_0 \leqq t \leqq t_1$ then (13) is **uniformly analytic** in the interval $t_0 \leqq t \leqq t_1$.

EXERCISES AND PROBLEMS II

1. Expand the function

$$x \sin x + s^2 + r \cos x + (r^2 + r^3) \sin^2 x$$

as a power series in $x - x^{(0)}$ and r where $x^{(0)} = \sin t$. The third powers of $x - x^{(0)}$ and r are sufficient to illustrate the process.

2. In the functions

$$f_1 = -\frac{(RI - E)[(r\, s_0/s)^2 + x_d x_q]}{L[(r\, s_0/s)^2 + x_{d'} x_q]},$$

$$g_1 = -\frac{2\, KP\, r^3 I^3 (x_d - x_{d'})[x_q^2 + (r\, s_0/s)]^2\, x_q s_0^2}{J I_0^2\, [(r\, s_0/s)^2 + x_d x_q]^3\, s^4 [(r\, s_0/s)^2 + x_{d'} x_q]},$$

$$f_2 = 0,$$

$$g_2 = -\frac{KP\, r\, I^2 [x_d^2 + (r\, s_0/s)^2]}{J\, s\, I_0^2\, [(r\, s_0/s)^2 + x_d x_q]^2},$$

let $s = s_0 e^{-z}$ and $I = \dfrac{E}{R} + I_1 e^{-y}$, where s, I, z, and y are variables and all other letters represent constants. Expand f_1, g_1, and g_2 in power series in $z_j - z_j^{(0)}$ and $y - y^{(0)}$ where $z_j^{(0)} = a_1 t + a_2 t^2$ and $y^{(0)} = b_1 t + b_2 t^2$. Two terms for each function are sufficient.

Instead of the single parameter r the system (2) may contain m parameters $r_1, r_2, \cdots, r_m$ in which an expansion is possible. In this situation Taylor's expansion formula, (13), can be extended to the $n + m$ variables $x_j - x_j^{(0)}$ and $r_1, r_2, \cdots, r_m$. It is, however, preferable to write $r_i = c_i r$ and obtain the expansions in $x_j - x_j^{(0)}$ and r alone. In the final answer $c_i r$ is then replaced by r_i, $i = 1, 2, \cdots, m$.

The parameter r may occur in g_i in two ways. In some terms (or expressions) it may occur simply, whereas in others it may occur in a highly complicated manner. When this is the situation, the parameter r can be set equal to r_0 in those terms in which it appears in a complicated way. The expansions can then be carried out in powers of $x_j - x_j^{(0)}$ and r, whereas r_0 is being treated as a mere constant. After the mathematical solution is completed, it must be remembered that the mathematical solution belongs to the physical or engineering problem only if $r = r_0$. Physics is a guide in the designation of the parameter as r or as r_0 in the expressions of the system. Understanding of the behavior of a physical system frequently diminishes its mathematical difficulties.

3·6. Synchronous Motor Operating Below Synchronous Speed with Field Unexcited. We shall now illustrate the method of analysis set forth in §§3·1–3·6 by the integration of the differential equations of performance of a synchronous machine operating as a reluctance-induction motor. The differential equation [6] of hunting of a synchronous motor of design such that the electrical transients, due to switching-on of the field voltage, do not appreciably affect the steady-state electrical torques, is

$$\frac{d^2\theta}{d\tau^2} + k(1 - b\cos 2\theta)\frac{d\theta}{d\tau} + r\sin 2\theta + \sin\theta = T, \qquad [15]$$

where

$$T = P_L/P_m, \quad a^2 = P_j/P_m, \quad r = P_r/P_m, \quad k = P_d/\sqrt{P_j P_m}$$

and where P_L, P_m, P_j, P_r, and P_d are constants defined elsewhere. The independent variable τ is given by $\tau = \lambda/a$ where λ is time in seconds.

Before the field voltage is switched on the motor may operate below synchronous speed as a reluctance-induction motor. The equation of performance [7] of such a motor is Eq. (15) with the term $\sin\theta$ deleted.

[6] For the derivation of Eq. (15) see H. E. Edgerton and P. Fourmarier, "The Pulling into Step of a Salient Pole Synchronous Motor," *Trans. A.I.E.E.*, **50** (June, 1931). For a more general differential system see D. R. Shoults, S. B. Crary, and A. H. Lauder "Pull-In-Characteristics of Synchronous Motors," *Elec. Eng.*, **54** (December, 1935).

[7] H. E. Edgerton and P. Fourmarier, *loc. cit.*

If change of dependent variable in (15) is made by the relation $2\theta = x$ then the required equation is

$$\frac{d^2x}{d\tau^2} + k(1 - b\cos x)\frac{dx}{d\tau} + 2r\sin x = 2T \qquad [16]$$

or, in normal form,

$$\begin{aligned} \frac{dx_1}{d\tau} &= x_2, \\ \frac{dx_2}{d\tau} &= 2T - kx_2 - 2r\sin x_1 + kc_1r\, x_2\cos x_1, \end{aligned} \qquad [17]$$

where $c_1r = b$ and $x_1 = x$.

Representative values of the parameters are

$$0 < b < 0.5, \quad 0.028 < k < 0.11 \quad (k \text{ for electrical degrees}),$$

$$0.3 < T < 0.8, \quad 0.25 < c < 0.50.$$

From physical considerations it is known that the solution of (17) consists of an oscillatory component superimposed upon a constant component of slip. Both the period and magnitude of the oscillatory component are unknown. However, it is known that both the period and magnitude of the oscillatory component are affected by and affect the constant component of slip. This physical situation frequently arises in certain types of engineering problems. Accordingly, the desired solution of (16) will illustrate, in addition to the principles of §§ 3·1–3·5, a method of solving this type of problem.

The procedure is as follows. First in (16) make the change of independent variable

$$\tau = (1 + \delta)t, \quad \delta = \delta_1 r + \delta_2 r^2 + \delta_3 r^3 + \cdots, \qquad [18]$$

where $\delta_1, \delta_2, \delta_3, \cdots$ are determined by subsequently imposed periodicity conditions. Next, (*a*) write the equation in t in the normal form, (*b*) expand $\sin x_1$ and $\cos x_1$ in power series in $x_1 - x_1^{(0)}$, (*c*) substitute

$$x_i(t) = x_i^{(0)}(t) + x_i^{(1)}(t)r + x_i^{(2)}(t)r^2 + \cdots \quad (i = 1, 2) \qquad [19]$$

in the differential equations resulting from steps (*a*), (*b*), and (*c*). Finally, in each of the two differential equations obtained thus far equate to zero the coefficients of each power of r. The final equations corresponding to Eqs. (14) are

$$\begin{cases} x_1^{(0)\prime} = x_2^{(0)}, \\ x_2^{(0)\prime} = -k\,x_2^{(0)} + 2T, \end{cases} \qquad [19a]$$

$$\begin{cases} x_1^{(1)\prime} = x_2^{(1)}, \\ x_2^{(1)\prime} = -k(\delta_1 x_2^{(0)} + x_2^{(1)} - c_1 x_2^{(0)} \cos x_1^{(0)}) - 2 \sin x_1^{(0)} + 4T\,\delta_1, \end{cases} \tag{19b}$$

$$\begin{cases} x_1^{(2)\prime} = x_2^{(2)}, \\ x_2^{(2)\prime} = -k[\delta_2 x_2^{(0)} + \delta_1(x_2^{(1)} - c_1 x_2^{(0)} \cos x_1^{(0)}) + x_2^{(2)} + x_1^{(1)} x_2^{(0)} c_1 \sin x_1^{(0)} \\ \qquad - x_2^{(1)} c_1 \cos x_1^{(0)}] - 2x_1^{(1)} \cos x_1^{(0)} - 4\delta_1 \sin x_1^{(0)} + 2T(\delta_1^2 + 2\delta_2), \end{cases} \tag{19c}$$

. . . .

The above sets of equations are now integrated sequentially. The general solution of (19*a*) is

$$\begin{aligned} x_1^{(0)} &= -\frac{A_0}{k} e^{-kt} + et + C_0, \\ x_2^{(0)} &= A_0\, e^{-kt} + e, \end{aligned} \tag{20}$$

where $e = 2T/k$. Only a steady-state solution is desired. Consequently, the initial conditions are chosen such that $A_0 = C_0 = 0$. Thus $x_1^{(0)}(0) = 0$, $x_2^{(0)}(0) = e$ and

$$x_1^{(0)}(t) = et, \quad x_2^{(0)}(t) = e. \tag{20a}$$

The substitution of (20*a*) in (19*b*) yields the differential equations

$$\begin{aligned} x_1^{(1)\prime} &= x_2^{(1)}, \\ x_2^{(1)\prime} &= -\,k(\delta_1 e + x_2^{(1)} - c_1 e \cos et) - 2 \sin et + 4T\,\delta_1, \end{aligned}$$

whose general solution is

$$\begin{aligned} x_1^{(1)} &= e\,\delta_1 t + \left[\frac{e(kc_1 + 2)\sin et - k(c_1 e^2 - 2)\cos et}{e(e^2 + k^2)}\right] - \frac{A_1 e^{-kt}}{k} + C_1 \\ x_2^{(1)} &= e\,\delta_1 + \left[\frac{k(e^2 c_1 - 2)\sin et + e(k^2 c_1 + 2)\cos et}{e^2 + k^2}\right] + A_1 e^{-kt}. \end{aligned} \tag{20b}$$

Choose $A_1 = C_1 = 0$. Then

$$x_1^{(1)}(0) = -\frac{k(e^2 c_1 - 2)}{e(e^2 + k^2)}, \quad x_2^{(1)}(0) = \frac{e(k^2 c_1 + 2)}{e^2 + k^2} + e\delta_1.$$

The value of δ_1 is now to be determined by the periodicity conditions. If (20*b*), with $A_1 = C_1 = 0$ are substituted in (19*c*) and if the solution of the resulting differential equations carried out with $\delta_1 \neq 0$ then terms of the form *t sin et, t cos et* appear. From physical considerations such terms cannot appear. Consequently, δ_1 must vanish.

Write (20*b*) in the form

$$x_2^{(1)} = D_1 \sin et + D_2 \cos et, \quad x_2^{(1)} = E_1 \sin et + E_2 \cos et.$$

Substituting these values of $x_1^{(1)}$ and $x_2^{(1)}$ in (19*c*) and integrating the resulting differential equations we have for the general solution

$$x_1^{(2)} = \left(-ke\delta_2 + 4T\delta_2 - \frac{ek}{2} c_1 D_1 + \frac{kc_1 E_2}{2} - D_2\right)\frac{t}{k} + \left[\frac{(k\beta - 2e\alpha)\sin 2et - (k\alpha + 2\beta e)\cos 2et}{2e(k^2 + 4e^2)}\right] - \frac{A_2}{k} e^{-kt} + C_2,$$

$$x_2^{(2)} = \frac{1}{k}(-ke\delta_2 + 4T\delta_2 - ek\, c_1 D_1 + kc_1 E_2 - D_2) + \left[\frac{(k\beta - 2e\alpha)\cos 2et + (k\alpha + 2\beta e)\sin 2et}{k^2 + 4e^2}\right] + A_2 e^{-kt},$$

where

$$\alpha = \frac{k^2 c_1(c_1 e^2 - 2) - (k^2 c_1 + 2)}{e^2 + k^2}, \quad \beta = \frac{ke^2 c_1(k^2 c_1 + 2) + k(c_1 e^2 - 2)}{e(e^2 + k^2)}.$$

The linear term in t in $x_1^{(2)}$ must vanish and consequently

$$\delta_2 = \frac{2 - c_1 e^2}{e^2(e^2 + k^2)}. \qquad [21]$$

If $A_2 = C_2 = 0$, the initial conditions are

$$x_1^{(2)}(0) = -\frac{(k\alpha + 2e\beta)}{2e(k^2 + 4e^2)}, \quad x_2^{(2)}(0) = \frac{(k\beta - 2e\alpha)}{(k^2 + 4e^2)}.$$

The entire solution as far as terms in r^2, when the relation $x = 2\theta$ is employed, is

$$\theta_1 = \frac{et}{2} + \frac{r}{2}\left[\frac{(k^2 c_1 + 2)e \sin et - k(e^2 c_1 - 2)\cos et}{e(e^2 + k^2)}\right] + \frac{r^2}{2}\left[\frac{(k\beta - 2e\alpha)\sin 2et - (k\alpha + 2\beta e)\cos 2et}{2e(k^2 + 4e^2)}\right] + \cdots \qquad [22]$$

$$\theta_2 = \frac{e}{2} + \frac{r}{2}\left[\frac{k(c_1 e^2 - 2)\sin et + e(c_1 k^2 + 2)\cos et}{e^2 + k^2}\right] + \frac{r^2}{2}\left[\frac{(k\beta - 2e\alpha)\cos 2et + (k\alpha + 2\beta e)\sin 2et}{k^2 + 4e^2}\right] + \cdots,$$

where $t = \dfrac{\lambda}{a(1 + \delta)} = \dfrac{\lambda}{a(1 + \delta_1 r + \delta_2 r^2 + \cdots)}$, λ being in seconds.

The periodic component of the slip is periodic of period $2\pi a(1 + \delta_1 r + \delta_2 r^2 + \cdots)$ in λ. The graphs of the angular displacement θ_1 and the

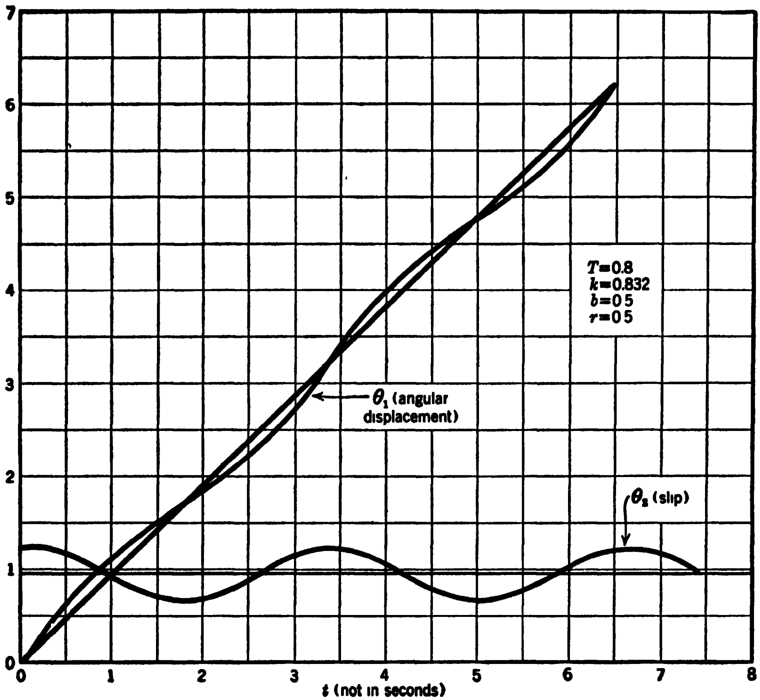

FIG. 3·2a. Slip and Angular Displacement for Rotor of Induction-reluctance Motor.

slip θ_2 are shown in Fig. 3·2a. The graph of the slip plotted against the displacement is shown in Fig. 3·2b.

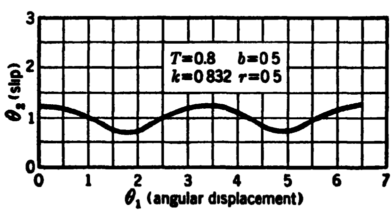

FIG. 3·2b. Slip Plotted Against Angular Displacement for Induction-reluctance Motor.

It is needless to state that the solutions for θ_1 and θ_2 can be continued to as high a power in r as is desired.

EXERCISES AND PROBLEMS III

1. Carry out the solution (22) of § 3·6 as far as the terms in r^3 and compute δ_3.

2. The differential equations of the field current I and speed s of a synchronous machine during dynamic braking are

$$\frac{dI}{dt} = -\frac{(RI - E)[(rs_0/s)^2 + x_d x_q]}{L[(rs_0/s)^2 + x_{d'} x_q]}$$
$$- \frac{2KPr^3I^3(x_d - x_{d'})[x_q^2 + (rs_0/s)^2]x_q s_0^2}{JI_0^2[(rs_0/s)^2 + x_d x_q]^3[(rs_0/s)^2 + x_{d'} x_q]s^4},$$
$$\frac{ds}{dt} = -\frac{KPrI^2}{JsI_0^2}\frac{[x_q^2 + (rs_0/s)^2]}{[(rs_0/s)^2 + x_d x_q]^2}.$$

The range of the constants for a typically small and typically large machine are

P = Rating of the machine kva = 15 or 400,
J = Moment of inertia, pounds ^{2} feet = 0.330 or 7.215,
s_0 = Initial speed, radians per second = 125.8 or 9.93,
t = Time in seconds,
K = Constant = 735.5,
s = Speed at any time, radians per second,
x_d = Direct synchronous reactance, per unit = 1.104 or 0.64,
x_q = Quadrature synchronous reactance, per unit = 0.767 or 0.46,
$x_{d'}$ = Direct transient reactance, per unit = 0.654 or 0.29,
r = Shorting resistance plus armature resistance, per unit = 0.682 or 0.277,
E = Field voltage, volts = 25 or 76.67,
I_0 = No load field current, amperes = 6.5 or 57.5,
I_1 = Jump in field current on short circuit, amperes = 5.82 or 32,
I = Field current at time t, amperes,
R = Field resistance, ohms = 3.54 or 0.802,
L = Field inductance, henrys = 0.512 or 0.67.

Inspection of speed curves and oscillograms of the field currents of typical machines suggests the change of dependent variables

$$s = s_0 e^{-z},$$
$$I = \frac{E}{R} + I_1 e^{-v},$$

where $s(0) = s_0$ and $I(0) = \dfrac{E}{R} + I_1$.

(*a*) Obtain the resulting differential equations in z and y.

(*b*) Note that the solution of

$$\frac{dy}{dt} = \frac{R}{L},$$

$$\frac{dz}{dt} = KPr \frac{\left[\frac{E}{R} + I_1 e^{-y}\right]^2 \left[x_q^2 + (re^z)^2\right]}{JI_0^2 s_0^2 e^{-2z} \left[x_d x_q + (re^z)^2\right]^2}$$

can be used as a generating solution. Obtain this solution $y = y^{(0)}(t)$, $z = z^{(0)}(t)$.

(*c*) Note that $x_d x_q - x_{d'} x_q$ and $\frac{2KPr^2}{JI_0^2 I_1} x_q(x_d - x_{d'})$ in the equations obtained in (*a*) are small. Call the first r_1 and the second r_2. Let $r_1 = c_1\mu$ and $r_2 = c_2\mu$.

(*d*) Expand, by Eqs. (13), the right members of the equations obtained in (*a*) in powers of $(y - y^{(0)})$, $(z - z^{(0)})$ and μ.

(*e*) To illustrate the method of this section (Type II) compute the solution, as far as and including the terms $y^{(1)}$ and $z^{(1)}$. (A better method of handling this particular problem is given in Sec. 9 of the present chapter.)

3·7. Convergence of the Solution (6). Thus far the solution (6) of (2) may be merely a formal solution and of no value. It remains to investigate the convergence of the series (6). It will be shown, in this article, by means of the well-known method of dominant functions that the series (6) converges for certain domains of r, $(x_j - x_j^{(0)})$, and t.

The gist of the method of dominant functions in establishing the existence of solutions of differential systems in normal form now follows. Some details in the method are left as exercises in problem set IV. The right members of the differential equations in question are expanded by (13) in series of the required type. Next, the right members of the differential equations are replaced by functions which, if expanded in series, are greater *term by term* (i.e., dominant) than the series of the right members of the given differential equations. *Moreover, the dominant series must be such that the dominant system can be integrated.* In general, certain restrictions will be imposed on the parameters of the dominant system in order that the solution of the dominant system converge. Since the solution of the dominant system converges, the solution (6) also converges because the solution (6) is less term by term than the dominant solution.

Explicitly, then suppose that the solution $x_i = x_i^{(0)}(t)$ of (3) has been found. If (3) are subtracted from (2) there results

$$\frac{d}{dt}(x_i - x_i^{(0)}) = f_i(x_j;t) - f_i(x_j^{(0)};t) + rg_i(x_j;r;t), \quad (i, j = 1, \cdots, n). \tag{23}$$

$$x_i - x_i^{(0)} = 0 \text{ for } t = t_0.$$

Suppose that the right members of (23) are expanded by (13) as power series in $x_j - x_j^{(0)}$ and r and then make the change of variables $x_i - x_i^{(0)} = X_i(t)$. The initial conditions for the new system in X_i are $X_i(t_0) = 0$. The right members of (23) are expansible in powers of $x_j - x_j^{(0)}$ and r within the region $|x_j - x_j^{(0)}| \leqq \rho_j$, $|r| \leqq \sigma$, and $t_0 \leqq t \leqq T$ provided f_i and g_i are analytic within this region. In engineering problems these conditions are always satisfied, if not over the complete interval $t_0 \leqq t \leqq T$, then at least over each of a finite number of subintervals into which (t_0, T) can be divided.

Let M_i be an upper bound of $|f_i(x_j;t) - f_i(x_j^{(0)};t) + \sigma_0 g_i(x_j;r;t)|$ in the region specified above. The quantity σ_0 satisfies the relation $0 < |r| < \sigma_0 < \sigma$. Let M be as large as any M_i. It is not difficult to see that (Ex. 1) the right members of (23), when expanded by (13), are dominated by the expansion of the right members of the equations

$$\frac{dX_i}{dt} = \frac{M\left[\dfrac{X_1 + \cdots + X_n}{\rho} + \dfrac{r}{\sigma_0}\right]\left[1 + \dfrac{X_1 + \cdots + X_n}{\rho} + \dfrac{r}{\sigma_0}\right]}{\left[1 - \dfrac{X_1 + \cdots + X_n}{\rho} - \dfrac{r}{\sigma_0}\right]}, \tag{24}$$

$$X_i(t_0) = 0,$$

where $\rho < \rho_j$. Since the right members in the n Eqs. (24) are all identical and since $X_i(t_0) = 0$ for $i = 1, 2, \cdots, n$ it follows that $X_1 = X_2 = \cdots = X_n$. Set $X_i = \dfrac{\rho}{n}\left(X - \dfrac{r}{\sigma_0}\right)$ in (24). Then X must satisfy the differential equation

$$\frac{dX}{dt} = \frac{nM}{\rho}\frac{X(1 + X)}{1 - X}, \tag{25}$$

$$X(t_0) = \frac{r}{\sigma_0}.$$

The solution of (25), subject to the initial conditions, is

$$X = -1 + \frac{Q(1 + r/\sigma_0)e^{-nM(t-t_0)/\rho}}{2r/\sigma_0},$$

where [26]

$$Q = 1 + r/\sigma_0 - \sqrt{(1 + r/\sigma_0)^2 - 4r/\sigma_0\, e^{nM(t-t_0)/\rho}}.$$

(See Ex. 2, problem set IV.)

The right member of (26) is expansible as a power series in r. Since the solution of (24) as a power series in r is unique, this solution is identical to the expansion of (26) as a power series in r.

It is next necessary to examine the region of convergence of the series resulting from (26). By the theory of functions the series in question converges interior to a circle whose center is zero and whose radius is the distance from the origin to the nearest singular point of the function X. The only finite singularities of X are the branch-points [8] defined by the equation

$$\left(1 + \frac{r}{\sigma_0}\right)^2 - \frac{4r}{\sigma_0} e^{nM(t-t_0)/\rho} = 0.$$

The two roots of this equation are

$$r = \sigma_0[-1 + 2e^{nM(t-t_0)/\rho} \pm \sqrt{(1 - 2e^{nM(t-t_0)/\rho})^2 - 1}];$$

the smaller of which is the one with the negative radical. From the smaller root

$$|r| < \frac{1 - \sqrt{(1 - e^{-nM(t-t_0)/\rho})}}{1 + \sqrt{(1 - e^{-nM(t-t_0)/\rho})}}\sigma_0. \qquad [26a]$$

(See Ex. 3.)

By the reasoning of the preceding paragraph the region of convergence in r of the solution of (24) as a power series in r is given by (26*a*).

The steps necessary to complete the proof of the existence of a solution of (2) in the form of (6) are as follows. (The details of the steps are left as Ex. 4.) Expand the right members of (23) as power series in $x_j - x_j^{(0)}$ and r. Substitute in these expansions

$$x_j - x_j^{(0)} = rx_j^{(1)} + r^2 x_j^{(2)} + \cdots. \qquad [26b]$$

Equate corresponding powers of r, obtaining a sequence of differential equations. Next, expand the right members of (24) as power series in X_j and r. Substitute in these expansions

$$X_j = rX_j^{(1)} + r^2 X_j^{(2)} + \cdots. \qquad [26c]$$

[8] See Vol. I, Chap. IV, or J. Pierpont, *Functions of a Complex Variable*, pp. 95, 235–238.

Equate corresponding powers of r, obtaining a sequence of differential equations. Show, by expressing the integrals of the two sequences of differential equations, that the series in the right member of (26*c*) is greater term by term than the series in the right member of (26*b*). The radius of convergence of (26*c*) is, however, given by inequality (26*a*) and consequently (26*b*) converges in the same domain. In fact, (26*b*) will usually converge for a larger value of r than indicated by (26*a*).

It is unfortunate that there exists no method in all mathematics of determining the *true* radius of convergence of (6) without first finding the series. By **true radius** is meant a value of r, say, r_0 such that for $r \leqq r_0$ the series converges and for $r > r_0$ the series diverges. This fact brings out an important engineering observation. In engineering investigations the value of r as given by (26*a*) is usually smaller than the value required in the problem under solution, but in important electrical problems there exist oscillograms and in important mechanical problems there exist very frequently differential analyzer solutions which may serve as answers or checks on analytical solutions and by these an idea of the convergence of (6) can frequently be ascertained. Often such electrical or analyzer solutions are of aid in the choice of the M_i and in the choice of dominant functions, Eqs. (24).

3·8. Differential Equations of Type I. The system of differential equations

$$\begin{aligned} x_i' &= rf_i(x_j;r;t), \quad (i, j = 1, \cdots, n) \\ x_i(t_0) &= a_i \end{aligned} \tag{27}$$

is known as a system of type I. Although systems of type II are of much wider applicability in engineering and applied science than are those of type I the latter are of considerable industrial importance. It is sufficient for our purpose if the functions $f_i(x_j;r;t)$ are analytic in x_j and r and continuous in t within the domain $|x_j - a_j| \leqq r_j, |r| \leqq \sigma$ for $t_0 \leqq t \leqq T$. For if these properties of the functions $f_i(x_j;r;t)$ do not exist for the entire interval of t for which the solution of (27) is desired they will exist at least over each of a finite number of subintervals into which (t_0, T) can be resolved.

Under a criterion subsequently stated (Eq. 29*a*) there exists a solution of (27) of the form

$$x_i(t) = a_i + x_i^{(1)}(t)r + x_i^{(2)}(t)r^2 + \cdots, \quad (i = 1, \cdots, n), \tag{28}$$

where the $x_i^j(t)$ are determined by solutions of Eqs. (29). To obtain (29) let $f_i(x_j;r;t)$ be expanded as power series in $x_j - a_j$ and r. Substitute the values of $x_j - a_j$ from (28) in Eqs. (27) after the right mem-

bers of (27) have been expanded. Equate like powers of r of the right- and left-hand members of the equations and obtain

$$\begin{aligned} \frac{dx_i^{(1)}}{dt} &= f_i(a_j;0;t) \quad (i, j = 1, \cdots, n), \\ \frac{dx_i^{(2)}}{dt} &= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} x_j^{(1)} + \frac{\partial f_i}{\partial r}, \\ &\cdots\cdots\cdots\cdots \end{aligned} \qquad [29]$$

Equations (29), like (14), can be integrated sequentially.

EXAMPLE. In the theory of the series non-linear circuit (§3·35) there is the following system of differential equations:

$$\frac{du}{dt} = -r(1 + 3b_3y_1^2 + 5b_5y_1^4)u \cos^2 (t + v),$$

$$\frac{dv}{dt} = r(1 + 3b_3y_1^2 + 5b_5y_1^4) \sin (t + v) \cos (t + v),$$

with the initial conditions $u(0) = e_0$, $v(0) = 0$, where r, b_3, and b_5 are constants; r is of the order of 0.1; u and v are dependent variables; and $y_1 = u \sin (t + v)$. From physical considerations in non-linear circuits and (28) there exists a solution of the form

$$u = e_0 + u_1r + u_2r^2 + \cdots, \quad v = v_1r + v_2r^2 + \cdots.$$

The quantities u_i, v_i are determined in §3·35.

Of course, series (28) do not converge for all values of r. Existence proofs, by means of dominant functions, yield theorems which specify conditions under which (28) is a solution of (27). One of the most useful of these theorems is: *Let T_1 be an arbitrary value of t such that $t_0 < T_1 \leqq T$. It is possible to determine a value of $|r|$, say, σ_0 such that* (28) *will converge for all values of r and t for which $|r| < \sigma_0$, $t_0 \leqq t \leqq T_1$.*

The above theorem follows as a consequence of inequality (29*a*). Inequality (29*a*) is established by means of dominant functions in much the same way that inequality (26*a*) was established. Let the functions $f_i(x_j;r;t)$ be analytic in x_j and r in the region $|x_j - a_j| \leqq \rho_j < \rho$, $|r| \leqq \sigma$. In choosing M_j the inequality $|r| < \sigma_0 < \sigma$ is satisfied. The common upper bound of $f_i(x_j;r;t)$ is denoted by M. The inequality corresponding to (26*a*) is

$$|r| < \frac{\sigma_0}{1 + 2nM\dfrac{\sigma_0}{\rho}(t - t_0)}. \qquad [29a]$$

The details of establishing (29*a*) are left as a problem for the student.

EXERCISES AND PROBLEMS IV

1. Show that the right members of (24), when expanded in powers of X_j and r, dominate the right members of (23) when expanded in series.

2. By separation of variables, solve (25) subject to the initial conditions $X(t_0) = r/\sigma_0$.

3. Obtain inequality (26*a*) from the equation which precedes it.

4. Fill in the analytical steps in the reasoning employed from inequality (26*a*) to the end of § 3·7.

5. By the method of Sec. (1) obtain a formal solution of the differential equation

$$\frac{dy}{dx} = x\left(\frac{1}{y} - 1\right) - rx,$$

with the initial condition $y(0) = y_0 < 1$. The ranges of variables in the physical problem are $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

6. Obtain a solution by the method of isoclines[9] of the differential equation of problem 5. Let the initial conditions and the ranges on the variables be the same as in problem 5. Determine the largest value of r for which the analytical and isocline solutions are in good agreement.

7. By the method of dominant functions obtain a value of r (say σ_0) in problem 5 such that a solution in the form of (6) converges for all r in the interval $0 \leqq r \leqq \sigma_0$.

(2)

Non-linear Systems by Variation of Parameters

The solutions of systems of non-linear equations are most conveniently carried out when the systems are expressed in normal form.

3·9. Generating Solution. Suppose the system reduced to the form given by (1). If any of the F_i consist of more than one term then (1) can be written in the form

$$\begin{aligned} x_i' &= f_i(x_j;t) + g_i(x_j;t), \\ x_i(t_0) &= a_i. \end{aligned} \qquad [30]$$

In general, the resolution of Eqs. (1) into the form (30) is not unique. The first part of the construction of a solution is to break up the F_i so that (*a*) $x_i' = f_i(x_j;t)$ represents the greater part of the system and (*b*) at the same time is solvable by either the elementary theory[10] or by the methods of §§ 3·1–3·8 or of § 3·13. Suppose then that a solution of

$$\begin{aligned} x_i' &= f_i(x_j;t), \quad (i = 1, 2, \cdots, n) \\ x_i(t_0) &= a_i, \end{aligned} \qquad [31]$$

[9] For method of isoclines, see Vol. I, p. 170.

[10] Any text on a first course in differential equations.

has been obtained and let it be denoted by

$$x_i = \varphi_i(a_1, a_2, \cdots, a_n; t) \cdot (i = 1, 2, \cdots, n) \cdot \qquad [32]$$

We may think of a_i as constants, since they are the n arbitrary constants of the solution, or as variables which in turn have constant values for some specified values. Suppose we consider them as variable parameters y_i and write the solution of (31) as

$$x_i = \varphi_i(y_1, y_2, \cdots, y_n; t). \qquad [33]$$

Equation (33) may be used as equations for change of variables in system (30). By the formula for total derivative,

$$\frac{dx_i}{dt} = \frac{\partial \varphi_i}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial \varphi_i}{\partial x_n}\frac{dx_n}{dt} + \frac{\partial \varphi_i}{\partial t},$$

equations (31) become

$$\begin{aligned} &\frac{\partial \varphi_1}{\partial y_1}\frac{dy_1}{dt} + \cdots + \frac{\partial \varphi_n}{\partial y_n}\frac{dy_1}{dt} + \frac{\partial \varphi_1}{\partial t} = f_1[\varphi_j(y_k,t);t] + g_1[\varphi_j(y_k,t);t], \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad [34] \\ &\frac{\partial \varphi_n}{\partial y_1}\frac{dy_1}{dt} + \cdots + \frac{\partial \varphi_n}{\partial y_n}\frac{dy_n}{dt} + \frac{\partial \varphi_n}{\partial t} = f_n[\varphi_j(y_k,t);t] + g_n[\varphi_j(y_k,t);t]. \end{aligned}$$

where $k, j = 1, 2, \cdots, n$.

Now $\dfrac{\partial \varphi_i}{\partial t}$ indicates the derivative of $\varphi_i(y_1, y_2, \cdots, y_n; t)$ with respect to t where t occurs *explicitly*, the y_j being considered as constants. Thus the functions $x_i = \varphi_i(y_1, y_2, \cdots, y_n; t)$ satisfy the equations

$$\frac{\partial \varphi_i}{\partial t} = f_i[\varphi_j(y_k,t); t]$$

and Eqs. (34) reduce to

$$\begin{aligned} &\frac{\partial \varphi_1}{\partial y_1}\frac{dy_1}{dt} + \cdots + \frac{\partial \varphi_1}{\partial y_n}\frac{dy_n}{dt} = g_1[\varphi_j(y_k,t);t], \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \qquad [35] \\ &\frac{\partial \varphi_n}{\partial y_1}\frac{dy_1}{dt} + \cdots + \frac{\partial \varphi_n}{\partial y_n}\frac{dy_n}{dt} = g_n[\varphi_j(y_k,t);t]. \end{aligned}$$

Equations (35) can be solved for $y_1', y_2', \cdots, y_n'$ under the same conditions that a set of n linear non-homogeneous equations in n unknowns can be solved. By Cramer's rule the solution is

$$\frac{dy_s}{dt} = \frac{g_1 A_{1s} + g_2 A_{2s} + \cdots + g_n A_{ns}}{\Delta(t)}, \quad (s = 1, 2, \cdots, n) \qquad [36]$$

where

$$\Delta(t) = \begin{vmatrix} \frac{\partial\varphi_1}{\partial y_1} & \cdots & \frac{\partial\varphi_1}{\partial y_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial\varphi_n}{\partial y_1} & \cdots & \frac{\partial\varphi_n}{\partial y_n} \end{vmatrix}$$

and A_{rs} is the cofactor of the element in the rth row and sth column of $\Delta(t)$.

Before discussing the nature of the solution of (36) it may be helpful to employ the method in the solution of an illustrative example.

3·10. Solution of Equation of Hunting. To illustrate the method of variation of parameters we shall solve Eq. (15). The normal form of (15) is

$$\begin{aligned} \varphi_1' &= \varphi_2, \\ \varphi_2' &= T - r\sin 2\varphi_1 - \sin\varphi_1 - k(1 - b\cos 2\varphi_1)\varphi_2, \end{aligned} \qquad [37]$$

where primes denote derivatives with respect to τ and where, upon correlation with Eq. (30),

$$\begin{aligned} f_1 &= \varphi_2, & g_1 &= 0, \\ f_2 &= T - r\sin 2\varphi_1 - \sin\varphi_1, & g_2 &= -k(1 - b\cos 2\varphi_1)\varphi_2. \end{aligned}$$

The initial conditions at the time of switching-on the exciter are given by Eqs. (22). Let these conditions be written $\varphi_1(t_0) = \theta_0$, and $\varphi_2(t_0) = y$.

The change of variables $\varphi_1 = \theta_1 + \theta_0$, $\varphi_2 = \theta_2$ in (37) and subsequent division of the second by the first of the resulting equations yield

$$\frac{d\theta_2}{d\theta_1} = \frac{T - r\sin 2(\theta_1 + \theta_0) - \sin(\theta_1 + \theta_0)}{\theta_2} - k[1 - b\cos 2(\theta_1 + \theta_0)].$$

The solution, which satisfies

$$\frac{d\theta_2}{d\theta_1} = \frac{T - r\ \sin\ 2(\theta_1 + \theta_0) - \sin\ (\theta_1 + \theta_0)}{\theta_2}$$

and the boundary conditions $\theta_1(t_0) = 0$, $\theta_2(t_0) = y$, is

$$(\theta_2)^2 = 2T\theta_1 + r\cos 2(\theta_1 + \theta_0) + 2\cos(\theta_1 + \theta_0) - r\cos 2\theta_0 - 2\cos\theta_0 + y^2, \qquad [38]$$

where y is an arbitrary constant or a new variable. Equation (38)

corresponds to (33) of the theory. The equation corresponding to (34) is

$$\frac{\partial\theta_2}{\partial y^2}\frac{dy^2}{d\theta_1} + \frac{\partial\theta_2}{\partial\theta_1} = \frac{T - r\sin 2(\theta_1+\theta_0) - \sin(\theta_1-\theta_0)}{\theta_2} - k[1 - b\cos 2(\theta_1+\theta_0)],$$

whence

$$\frac{dy^2}{d\theta_1} = -2k[1 - b\cos 2(\theta_1+\theta_0)]\theta_2, \tag{39}$$

where θ_2^2 is given by (38). To obtain θ_2 from (38) it is necessary to extract the square root of the right member of (38). To do this expand the right member of (38) as a power series in θ_1 obtaining

$$e_0 + e_1\theta_1 + e_2\theta_1^2 + \cdots,$$

where the e_i are known constants. Set

$$e_0 + e_1\theta_1 + e_2\theta_1^2 + \cdots = (f_0 + f_1\theta_1 + f_2\theta_1^2 + \cdots)^2,$$

where the f_i are determined by squaring the right member and equating like powers of θ_1 on the two sides of the equation. Equation (39) now is

$$\frac{dy^2}{d\theta_1} = -2k[1 - b\cos 2(\theta_1+\theta_0)][f_0 + f_1\theta_1 + f_2\theta_1^2 + \cdots]. \tag{40}$$

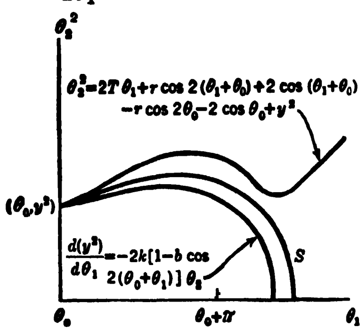

FIG. 3·3. Pulling-into-step of Synchronous Machine.

By reference to § 3·6 evidently $k < 1$. The above differential equation is of type I (§ 3·8) and there thus exists a solution as a power series in k.

Since the derivative in (40) is always negative the quantity y^2 is always a decreasing function until the slip θ_2 is zero. The graph of (38) for y^2 a constant is shown in Fig. 3·3. The curve labeled S is the complete solution of (37) up to the first time that $\theta_2 = 0$.

EXERCISES AND PROBLEMS V

1. By the method of variation of parameters, solve the following differential equations.

(a) $x' = ax + e^{at}$,

(b) $x' = \dfrac{2t}{t^2+1}x + 2t(t^2+1)$,

(c) $x'\cos t + x\sin t = 1$,

(d) $x' = ax + e^{mt}$,

(e) $(p^2 - 2p + 3)x = e^t \sin 2t$,

(f) $x' = (\sin t)x + \cos t$.

2. Solve, by the method of variation of parameters, the differential equation $x' + P(t)x = Q(t)$, where $P(t)$ and $Q(t)$ are functions of t.

3. Solve, by the method of variation of parameters, the differential equation $(p^2 + ap + b)x = f(t)$, where a and b are constants and $f(t)$ is a function of t.

4. Solve the differential equation $s'' + k^2s = -g$, where k and g are constants.

5. Obtain, by the method of variation of parameters, the solution of the system

$$\begin{aligned} z_{11}(p)i_1 + \cdots + z_{1n}(p)i_n &= f_1(t), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &\quad \cdot \quad \cdot \quad \cdot \\ z_{n1}(p)i_1 + \cdots + z_{nn}(p)i_n &= f_n(t), \end{aligned}$$

where $z_{rs}(p) = L_{rs}p^2 + R_{rs}p + C_{rs}$ and L_{rs}, R_{rs}, C_{rs}, are constants.

6. Solve Eq. (40) as a power series in k inclusive of the term in k^2.

7. The equation of a simple pendulum, where the damping force is proportional to the square of angular speed, is

$$\theta'' \pm k(\theta')^2 = -\frac{g}{l}\sin\theta.$$

The algebraic sign depends upon the direction of motion. By the method of variation of parameters, solve the differential equation for θ'.

8. The elastic law for a certain non-linear spring is $f = kx + rx^3$, where x is the elongation and r is small relative to k. The differential equation of motion of a mass m attached to the spring is

$$m\frac{d^2x}{dt^2} + kx + rx^3 = 0.$$

If $x(0) = 0$, $x'(0) = a$, find a periodic solution of the differential equation and determine approximately the period of the solution.

HINT: The solution is by the method of §3·6. Let $t = (1 + \delta)\tau$ and write the differential equation

$$m\frac{d^2x}{d\tau^2} + (1 + \delta)^2kx + r(1 + \delta)^2x^3 = 0,$$

where $\delta = \delta_1r + \delta_2r^2 + \cdots$. Substitution of $x = x^{(0)} + x^{(1)}r + \cdots$ in the differential equation and the equating of like powers of r yields a sequence of linear differential equations. The imposed condition of periodicity determines sequentially $\delta_1, \delta_2, \cdots$.

9. Solve problem (8) when a periodic force $F = E\sin nt$ acts on the mass m and the differential equation becomes

$$m\frac{d^2x}{dt^2} + kx + rx^3 = E\sin nt.$$

10. The differential equation

$$\frac{du}{dt} = \frac{b\sin\omega t}{u} - a$$

arises in the study of muffle chamber discharge. The variable u is a measure of the pressure within the muffler. Figure 3·4b shows the nature of the variation of u in the operation of the muffler. At the point in the cycle when $\omega t = \omega t_0 = 125°$ there is a discharge into the cylinder at the intake port A. The pressure then builds up to

the value v_2 at $\omega t = 180°$ and then decreases, by exhaust through the port B, to the value v_0 according to the equation $\frac{du}{dt} = -a$.

The problem is to determine v_0, the minimum positive pressure, so that $v_1 = v_2$ in the steady-state operation of the muffle chamber. The unknowns are v_0, v_1, v_2. The

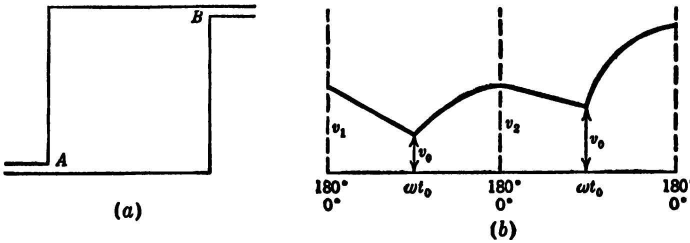

FIG. 3·4. Pressure in Muffler.

constants of the differential equation for a typical machine are $\omega = 177.8$, $a = -1009$, $b = 1.89 \times 10^3$.

HINT: If the independent variable is changed from t to θ by the equation $t = k\theta$ where $k = 1/b$, then the differential equation becomes

$$\frac{du}{d\theta} = \frac{\sin k\omega\theta}{u} - \frac{a}{b} = \frac{\sin k\omega\theta}{u} - r.$$

This equation holds for $\omega t_0 \leqq k\theta\omega \leqq 180°$ and the equation $\frac{du}{d\theta} = -r$ is valid for $0 \leqq k\theta\omega \leqq \omega t_0$. Since r is small, set

$$u = u_0 + u_1 r + u_2 r^2 + \cdots$$

in $uu' = \sin k\omega\theta - ru$ and get, by equating like powers of r,

$$\begin{aligned} u_0u_0' &= \sin k\omega\theta, \\ u_0u_1' + u_0'u_1 &= -u_0, \\ u_0u_2' + u_0'u_2 &= -u_1u_1' - u_1, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

The solution of the first equation of this sequence, subject to the condition $u_0 = v_0$ for $\theta = \theta_0$, is the generating function

$$u_0 = \left[v_0^2 + \frac{2}{k\omega}(\cos k\omega\theta_0 - \cos k\omega\theta)\right]^{1/2}.$$

A simple function approximating the generating function is

$$u_0 = v_0 + a_1 k\omega(\theta - \theta_0)^{1/2},$$

where a_1 is so determined that the values of u_0 as given by the last two equations are identical at $k\omega\theta = 180°$.

Continue the solution as far as the term in r and find the value of v_0, subject to the condition $v_1 = v_2$.

11. With this value of v_0 integrate the differential equation by means of the method of isoclines until a value of v_0 is obtained which yields $v_1 = v_2$ to two decimal places.

12. Obtain a better analytical solution of problem 10 than the one suggested above.

3·11. General Theory Resumed. If in Eqs. (36) $\Delta(t)$ vanishes for values of t for which the solution of the system is desired, then difficulties are introduced into the solution and other methods may then be preferable.

In some problems the $g_i[\varphi_j(y_k;t)]$ of (35) may vary either rapidly or slowly due to the presence of t explicitly, but slowly due to slowly changing y_k. In this case the y_k can be replaced by constant values $y_k(t_0)$ in the $g_i[\varphi_j(y_k;t)]$ without modifying appreciably the solution. The right members of (35) are then explicit functions of the time, and the difficulty of the problem is greatly reduced.

This naturally raises the need for a criterion for the possibility of setting $y_k = y_k(t_1)$ in the g_i. (*a*) Frequently, from engineering knowledge and Eqs. (33) the range of y_k is known. The y_k may then be set first equal to their least values and then to their greatest values in their domain and (35) solved for both sets. If the two solutions are approximately equal then either solution is satisfactory. (*b*) Recourse may be had to the differential analyzer or numerical integration for representative values of the parameters involved. These numerical solutions will serve as a check on the substitution in question.

3·12. Analytic Implicit Function Theory. When the generating functions are implicit functions of the dependent variables it may be possible to express the dependent variables as explicit functions of the independent variables. When the generating functions are implicit functions of both dependent and independent variables it may be possible to express the dependent variables as explicit functions of the independent variables or as an explicit function of some parameter r.

The reversion of series is the simplest case of the theory desired. Suppose that the generating function $F(x, t) = 0$ is of the form $t = f(x)$, where $f(x)$ is an analytic function of x in the interval $|x - a| \leqq \rho$. Then $f(x)$ is expansible in the convergent Taylor's series

$$t = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots . \qquad [41]$$

It is supposed further that $a_1 \neq 0$. Then $(x - a)$ can be developed as a power series in $(t - a_0)$ which is convergent for $t - a_0$ sufficiently

small. If $(t - a_0)/a_1$, $x - a$, and a_i/a_1 are replaced respectively by T, X, and A_i the series is

$$T = X + A_2X^2 + A_3X^3 + \cdots. \qquad [42]$$

Assume that

$$X = T + b_2T^2 + b_3T^3 + \cdots \qquad [43]$$

and substitute this value of X in the series for T. If the resulting series is rearranged according to powers of T and if coefficients of corresponding powers of T on both sides of the series are equated, the following relations are obtained.

$$\begin{aligned} b_2 &= -A_2 = -\frac{a_2}{a_1}, \\ b_3 &= 2A_2^2 - A_3 = \frac{2a_2^2}{a_1^2} - \frac{a_3}{a_1}, \\ b_4 &= -5A_2^3 + 5A_2A_3 - A_4 = -5\left(\frac{a_2}{a_1}\right)^3 + \frac{5a_2a_3}{a_1^2} - \frac{a_4}{a_1}, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad [44]$$

Finally

$$X = x - a = \frac{(t - a_0)}{a_1} - \frac{a_2}{a_1}\left(\frac{t - a_0}{a_1}\right)^2 + \frac{2a_2^2 - a_3a_1}{a_1^2}\left(\frac{t - a_0}{a_1}\right)^3 + \cdots. \qquad [45]$$

The coefficients $b_2, b_3, b_4, \cdots$ have been computed to the thirteenth term.[11] The series (45) can be tested by the usual methods. If $a_1 = 0$ it is still possible, under certain conditions, to reverse the given series.

Suppose next that the n generating functions are

$$\begin{aligned} F_1(x_1, \cdots, x_n; r) &= 0, \\ &\cdots\cdots\cdots \\ F_n(x_1, \cdots, x_n; r) &= 0, \end{aligned} \qquad [46]$$

where the functions F_i are analytic in the region $|x_j - a_j| \leqq \rho_j$ and $0 < r \leqq r_0$. The functions F_i are expansible in powers of $x_i - a_i$ by (13). It is further supposed that $x_i = a_i$ and $r = 0$ satisfy Eqs. (46), i.e., that the curve defined by (46) passes through the point $(a_1, a_2 \cdots, a_n, 0)$. There is no loss in generality in choosing the origin of coordinates so that $a_1 = a_2 = \cdots = a_n = 0$. The F_j are then analytic for $|x_j| \leqq \rho_j$ and $0 < r \leqq r_0$.

[11] C. E. van Orstrand, "Reversion of Power Series," *Phil. Mag.* [6], 19 (1910).

The Maclaurin expansions of (46) by (13) are

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= c_1 r + p_1(x_1, \cdots, x_n; r), \\ \cdots & \cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= c_n r + p_n(x_1, \cdots, x_n; r), \end{aligned} \qquad [47]$$

where a_{ij} and c_i are constants and p_i are power series in x_j containing no terms lower than the second degree in x_j and r.

If the determinant

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

does not vanish, Eqs. (47) can be solved for $x_1, \cdots, x_n$ in terms of the right members of these equations. Let the solution be

$$x_i = a_i r + \varphi_i(x_j; r) \quad (i, j = 1, 2, \cdots, n) \qquad [48]$$

where the φ_i are power series in x_j and r of degree two or greater.

To obtain a formal solution of (47) in powers of r substitute

$$x_i = a_i^{(1)} r + a_i^{(2)} r^2 + \cdots \quad (i = 1, 2, \cdots, n) \qquad [49]$$

in (48) and equate coefficients of corresponding powers of r on both sides of each equation. These equalities determine uniquely the coefficients $a_i^{(1)}, a_i^{(2)}, \cdots$ in (49).

It can be shown,[12] by means of dominant functions §3·7, that the series (49) converge for all values of r for which

$$|r| < \frac{r_0'}{1 + \dfrac{4n}{\rho} M \left[1 + \dfrac{n}{\rho} M\right]}, \qquad [50]$$

where $r_0' < r_0$, $\rho < \rho_j$, $n =$ the number of equations, and M is a constant upper bound of $|\varphi_i(x_j; r)|$ for $|x_j| \leqq \rho_j$ and $|r| \leqq r_0$. However, the domain of convergence of (48) is usually greater than that given by inequality (50).

Those cases for which the determinant in a_{ij} vanishes are discussed elsewhere.[13]

[12] F. R. Moulton, *Differential Equations*, p. 81.

[13] F. R. Moulton, *Periodic Orbits*, Carnegie Publication 161, Chap. I; W. D. MacMillan, *Mathematische Annalen*, **72**, 157–202.

EXERCISES VI

1. Reverse the following series so that the results contain terms in the fifth power of t.

(a) $t = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$

(b) $t = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

(c) $t = 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$.

2. Obtain the solution of

$$a_{10}x + a_{01}r + a_{20}x^2 + a_{11}xr + a_{02}r^2 + \cdots = 0$$

for x as a power series in r to terms in r^3.

3. Solve the equation

$$0 = 2x - r + \tfrac{1}{2}x^2 + \tfrac{1}{3}xr + \tfrac{1}{4}r^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^2r + \tfrac{1}{2}xr^2 + \tfrac{1}{5}r^3 + \cdots$$

for x as a power series in r to terms in r^3.

4. Solve the equations

$$\sin x_1x_2 + r + x_1 + x_2 = 0,$$
$$e^{x_1+x_2+r} + x_1 - x_2 + 3r + e^{x_1+x_2} - 2 = 0$$

as a power series in r as far as the terms in r^2.

3·13. Generating Functions in Series Form; Additional Observations on Convergence. It may be impossible to resolve (1) into the system of (2) such that the solution of system (3) shall resemble the solution of Eqs. (2) and at the same time be integrable by the elementary methods explained in a first course in differential equations. Moreover, it may be impossible to introduce into (3) a suitable parameter in powers of which a series solution can be obtained. Under these circumstances and as a last resort a solution as a power series in the independent variable may be attempted. For the technique of power series solutions in the *independent variable*, the reader is referred elsewhere.[14] In engineering work, power series solutions in the independent variables very frequently fail due to lack of convergence or due to complexity. In both cases the evident properties of the solution are lost.

The methods of Sec. 1–2 are methods of great power. Even more difficult problems are solvable when both methods are used sequentially in either order and with any number of repeated applications of the methods.

Additional observations on the question of convergence may be of value. If, in (26*a*), $t - t_0$ is sufficiently small then a value of r always

[14] Any text on a first course in differential equations.

exists for which the solution given by (6) converges. In many engineering problems a solution is necessary for all values of the time and not for the time in a restricted interval. If $t - t_0 = \infty$ in (26a), then $r = 0$. This is no cause for alarm, because (26a) does not give the true radius of convergence. In fact, the value of r can be much larger than zero and the series converge in the infinite interval $t - t_0$.

Another observation is important. It may be known from the physics of a problem that a periodic solution exists. An example is problem 8, set V. If the substitution $t = (1 + \delta)\tau$ is *not* made but if x is replaced by $x = x^{(0)}(t) + x^{(1)}(t)r + \cdots$ and if the solution of the differential equation is reduced, in the usual manner, to the solution of a sequence of linear differential equations, then it will be found that powers of t will appear in the solution. This solution is valid for r and T sufficiently small in the interval $t_0 < t \leqq T$. This solution resulted from an attempt to force on the differential equation a solution whose period is the period of the solution of the equation $m\dfrac{d^2x}{dt^2} + kx = 0$. In the application of the methods of Sec. 1-2, skill must frequently be employed if suitable solutions are to be found. The physics underlying the problem is the guide in finding suitable solutions.

PROBLEMS VII

1. Solve the differential equation (16) or (17) by substituting $x = x^{(0)}(\tau) + x^{(1)}(\tau)r + x^{(2)}(\tau)r^{(2)} + \cdots$ directly in the differential equation. The computation of $x^{(0)}(\tau)$, $x^{(1)}(\tau)$, and $x^{(2)}(\tau)$ are sufficient.

2. The differential equation of the free torsional vibrations of a flywheel with variable moment of inertia is

$$\frac{d}{dt}(I\dot{\theta}) + k\theta = 0,$$

where k is the torque constant of the shaft on which the flywheel is mounted. Let I be represented by $I = I_0(1 + r\sin\omega t)$, where r is small relative to unity. The differential equation then is

$$I_0\ddot{\theta} + \frac{I_0 r\omega\cos\omega t\,\dot{\theta}}{1 + r\sin\omega t} + \frac{k\theta}{1 + r\sin\omega t} = 0,$$

or, if damping be neglected and obvious approximations made,

$$I_0\ddot{\theta} + k(1 - r\sin\omega t)\theta = 0.$$

Obtain a solution of the differential equation by the following steps:

(a) Use as a generating function $\theta = A\cos nt + B\sin nt$, $n^2 = k/I$, which is the general solution of

$$\theta_1' = \theta_2, \quad \theta_2' = -k\theta_1.$$

(*b*) Employ the method of variation of parameters.

(*c*) Express A and B by the equations

$$A = A_0(t) + A_1(t)r + \cdots, \quad B = B_0(t) + B_1(t)r + \cdots$$

and complete the solution for A and B as power series in r.

(*d*) Note that the solution is for the interval $t_0 < t \leqq T$ and that it contains powers of t.

(*e*) Try to obtain, by the methods already explained, a periodic solution for all finite values of t.

(3)

Solutions of Systems by Method of Successive Integrations

The method of successive integrations is frequently of value in obtaining an approximate solution of a system of differential equations. Moreover, it is basic in the development of the matrix methods of Sec. 4 and in the integral equation method of Sec. 9 of this chapter.

3·14. Approximating Sequences. Let the system of differential equations be reduced to the normal form

$$\begin{aligned} x_i' &= f_i(x_j;t), \quad (i, j = 1, 2, \cdots, n) \\ x_i(t_0) &= a_i, \end{aligned} \qquad [51]$$

where the properties of the functions f_i are specified in a closed region

$$-r_i \leqq x_i - a_i \leqq r_i, \quad 0 \leqq t - t_0 \leqq \rho, \quad (i = 1, 2, \cdots, n). \qquad [52]$$

The functions f_i of this section may be more general functions than the f_i of Sec. 1. However, engineering functions possess, at least in a finite interval, the properties specified for the f_i of Sec. 1. An engineering problem can be solved for each of the finite intervals over which the f_i are analytic. It will be sufficient for the validity of the method of this section to assign to the f_i here employed, the properties of the f_i of Sec. 1.

The method consists formally in determining sequences of functions $x_i^{(1)}, x_i^{(2)}, \cdots, (i = 1, 2, \cdots, n)$ the limit of which constitutes the solution of (51). The sequences are defined by the equations

$$\begin{aligned} x_i^{(1)} &= a_i + \int_{t_0}^{t} f_i(a_j;t)dt, \quad (i = 1, 2, \cdots, n), \\ x_i^{(2)} &= a_i + \int_{t_0}^{t} f_i(x_j^{(1)};t)dt, \\ &\cdots\cdots\cdots\cdots \\ x_i^{(k)} &= a_i + \int_{t_0}^{t} f_i(x_j^{(k-1)};t)dt, \\ &\cdots\cdots\cdots\cdots \end{aligned} \qquad [53]$$

The proof[15] exists that the sequences defined by (53) possess limits and that these limits constitute the solution of (51).

EXAMPLE. Solve by the method of successive approximations the system

$$x_1' = 2x_1,$$

$$x_2' = x_1 + x_2,$$

subject to the initial conditions $x_i(0) = a_i$.

Let $\int_0^t (\quad)dt$ be denoted by Q and $x_i^{(k)}$ by x_i^k. Then the sequences corresponding to (53) are

$$\begin{cases} x_1^1 = a_1 + \int_0^t 2a_1 dt = a_1 + 2Qa_1\,, \\ x_1^2 = a_2 + \int_0^t (a_1 + a_2)dt = a_2 + (a_1 + a_2)t, \end{cases}$$

$$\begin{cases} x_1^2 = a_1 + 2Q(a_1 + 2Qa_1) = a_1(1 + 2t + 2t^2), \\ x_2^2 = a_2 + Q[(a_1 + a_2) + (3a_1 + a_2)t] + 2a_1t^2, \end{cases}$$

$$\begin{cases} x_1^3 = a_1 + 2Qa_1(1 + 2t + 2t^2) = a_1(1 + 2t + 2t^2 + \tfrac{4}{3}t^3), \\ x_2^3 = a_2 + [(a_1 + a_2)t + \tfrac{1}{2}(3a_1 + a_2)t^2 + \tfrac{1}{6}(7a_1 + a_2)t^3], \end{cases} \qquad [54]$$

$$\begin{cases} x_1^4 = a_1(1 + 2t + 2t^2 + \tfrac{4}{3}t^3 + \tfrac{8}{12}t^4 \cdots) = a_1e^{2t}, \\ x_2^4 = a_2e^t + a_1(t + \tfrac{3}{2}t^2 + \tfrac{7}{6}t^3 + \tfrac{15}{24}t^4 + \cdots), \end{cases}$$

$$= a_2e^t + a_1[(1 + 2t + 2t^2 + \tfrac{4}{3}t^3 + \cdots) - (1 + t + \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 + \cdots)]$$

$$= a_2e^t + a_1(e^{2t} - e^t).$$

The solution by inspection is $x_1 = a_1e^{2t}$, $x_2 = a_2e^t + a_1(e^{2t} - e^t)$.

3·15. Use of Approximate Solution. The principal weakness of the method of this section is the slow convergence, in many engineering problems, of the sequences defined by (53). The successive approximations, after the second or third step, may become too cumbersome. This difficulty is sometimes avoided if an approximate solution $x_i = a_i + \varphi_i(t)$ is known. It can be rigorously shown that the limit of the sequences (53), where the first Eqs. in (53) are taken to be

$$x_i^{(1)} = a_i + \int_{t_0}^t f_i[a_j + \varphi_j(t);t]dt \quad (i = 1, 2, \cdots, n) \qquad [54a]$$

[15] E. L. Ince, *Ordinary Differential Equations*, p. 63; F. R. Moulton, *Differential Equations*, p. 189.

and where $a_j + \varphi_j(t)$ are continuous functions of t in the region defined by (52), is the solution of (51). This device is employed in Sec. 9 of this chapter.

EXERCISES VIII

Solve by the method of successive integrations:

1. $x_1' = -x_1 - 2x_2, \quad x_2' = x_1 - x_2.$
2. $x\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) + x^2y = 0$, subject to initial conditions $y(0) = 1$, $y'(0) = 0$.
3. $x' = ax.$
4. $\dfrac{dy}{dx} - Ry = S$, where R and S are functions of x.
5. $x'' + ax' + bx = \mathrm{e}$ where $a^2 - 4ab < 0$.
6. $x'' + Rx' = \mathrm{S}$ where R and S are functions of t.

(4)

Solutions of Systems by Matrix Methods

The method of solution by matrices is largely the method of successive integrations recast in matrix notation. However, it differs in the following respects. The matrix method is more convenient than the method of successive integrations. The method of successive integrations is applicable to both linear and non-linear systems. The method of matrices is, at present, adapted only to linear equations. When the matrix method is applied to systems of equations possessing coefficients which are functions of the independent variable it yields a convenient method of numerical integration superior to the method explained in Chap. I, Vol. I. The method of this section does not pertain to non-linear systems. Before explaining the method it is necessary to state and illustrate certain theorems regarding matrices in addition to those theorems of Chap. II.

3·16. Certain Definitions and Theorems on Matrices. The equation

$$\varphi(\lambda) = |\lambda I - [a]| = 0;$$

where $[a]$ is an n-rowed square matrix whose elements a_{ij} are constants, I is unit matrix, λ is a parameter, and $|\lambda I - [a]|$ is a determinant, is called the **characteristic equation** of $[a]$. The n roots of the characteristic equation are called the **latent roots** of $[a]$.

The following theorem is an important theorem of matrix theory. *If $[a]$ is a square matrix and $\varphi(\lambda) = 0$ is its characteristic equation then* $\varphi([a]) = 0$.

In the theory of functions of a complex variable the definitions of the calculus were extended to the case where the independent variable was the complex variable $z = x + iy$. It is here desirable to extend the definitions of functions so that the independent variable is the matrix $[u]$. The following theorem is basic in these definitions. If $P([u])$ is any polynomial of the square matrix $[u]$, whose latent roots are λ_1, λ_2, $\cdots$, λ_n then

$$P([u]) = \sum_{r=1}^{n} P(\lambda_r)[Z_r], \qquad [55]$$

where the matrix $[Z_r]$ is

$$[Z_r] = \frac{\prod\limits_{s\neq r} (\lambda_s I - [u])}{\prod\limits_{s\neq r} (\lambda_s - \lambda_r)}.$$

For a proof of this theorem see Ex. 7.

EXAMPLE. By means of Eq. (55) express

$$P([u]) = [u]^2 + 3[u] + I, \text{where } [u] = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \text{ as a matrix.}$$

In this case the characteristic equation reduces to $(\lambda - 1)(\lambda - 2) = 0$ and the latent roots are $\lambda_1 = 1, \lambda_2 = 2$. Then

$$[Z_1] = \frac{(2)I - \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}}{2 - 1}, \quad [Z_2] = \frac{(1)I - \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}}{1 - 2}$$

$$P([u]) = \sum_{r=1}^{2} P(\lambda_r)[Z_r] = 5\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - 11\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 6 & 11 \end{bmatrix}.$$

This result is easily checked by squaring $[u]$ and adding to the square $3[u] + I$.

3·17. Functions of a Matrix. Since a polynomial $P(x)$ can be used to approximate a function $f(x)$ of elementary mathematics, Eq. (55) with $P([u])$ replaced by $f([u])$ can be used as the definition of a function of a matrix.

EXAMPLE. Express $f([u]) = e^{[u]}$, where $[u] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, as a matrix.

The characteristic equation is $(\lambda - 1)(\lambda - 2) = 0$ and the latent roots

are $\lambda_1 = 1, \lambda_2 = 2$. The expressions for $[Z_r]$ are

$$[Z_1] = \frac{2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}}{\lambda_2 - \lambda_1} = \frac{\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}}{2 - 1} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},$$

$$[Z_2] = \frac{1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}}{\lambda_1 - \lambda_2} = \frac{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}}{1 - 2} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

Equation (55) gives

$$e^{[u]} = e^1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + e^2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}e^1 & 0\\0 & e^2\end{bmatrix}.$$

3·18. Derivative and Integral of a Matrix. The derivative and integral of a matrix are defined by the equations

$$\frac{d[u]}{dt} = \begin{bmatrix}\dfrac{du_{11}}{dt} & \cdots & \dfrac{du_{1n}}{dt}\\ \cdot & \cdot\quad\cdot\quad\cdot & \cdot\\ \dfrac{du_{n1}}{dt} & \cdots & \dfrac{du_{nn}}{dt}\end{bmatrix}, \qquad \int_{t_0}^{t}([u])dt = \begin{bmatrix}Qu_{11} & \cdots & Qu_{1n}\\ \cdot & \cdot\quad\cdot\quad\cdot & \cdot\\ Qu_{n1} & \cdots & Qu_{nn}\end{bmatrix},$$

where $Qu_{ij} = \int_{t_0}^{t}(u_{ij})\,dt$.

EXERCISES IX

1. By means of Eq. (55), express as a matrix $\sin [u]$, where

$$[u] = \begin{bmatrix}1 & 0 & 0\\2 & 2 & 0\\3 & 4 & 3\end{bmatrix}.$$

2. Evaluate $e^{[u]}$, approximately where $[u] = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, by means of the series

$$e^{[u]} = I + [u] + \tfrac{1}{2}[u]^2 + \cdots.$$

3. Verify the first theorem of 3·16 for $[a] = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$.

4. Prove the first theorem of 3·16 for $[a]$ an n-rowed square matrix.

5. By means of Eq. (55), express as a matrix $\tan^{-1}[u]$, where

$$[u] = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}.$$

6. Express $\log [u]$ as a matrix, where $[u]$ has the same value as in Ex. 5.

7. Establish Eq. (55) by filling in the details in the following outline of a proof.

Let $p([u])$ be any polynomial of degree m in the square matrix $[u]$. Let $\varphi(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$ be the characteristic equation of $[u]$. From $\varphi(\lambda) = 0$ we have

$$\lambda^n = -a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots - a_n,$$

$$\lambda^{n+1} = -a_1\lambda^n - a_2\lambda^{n-1} - \cdots - a_n\lambda.$$

Substituting the value of λ^n from the first equation in the last we have

$$\lambda^{n+1} = -a_1(-a_1\lambda^{n-1} - \cdots - a_n) - \cdots - a_n\lambda.$$

By a continuation of this process it is possible to express $p(\lambda)$, a polynomial of degree m, as a polynomial $P(\lambda) = P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots + P_n$. But $[u]$ satisfies its own characteristic equation and thus all the relations written for λ are valid when λ is replaced by $[u]$.

Lagrange's interpolation formula [16] for the n points $[a_1, P(a_1)]$, $[a_2, P(a_2)]$, $\cdots$, $[a_n, P(a_n)]$ is

$$P(\lambda) = P(a_1)\frac{(\lambda - a_2)(\lambda - a_3)\cdots(\lambda - a_n)}{(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n)} + \cdots +$$

$$P(a_n)\frac{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)\cdots(\lambda - a_{n-1})}{(a_n - a_1)(a_n - a_2)\cdots(a_n - a_{n-1})},$$

where $a_1, a_2 \cdots, a_n$ are arbitrary. If $a_n = \lambda_r$, where λ_r $(r = 1, 2, \cdots, n)$ are the latent roots of $[u]$, then

$$p([u]) = P([u]) = \sum_{r=1}^{n} P(\lambda_r)\,[Z_r],$$

where $[Z_r]$ is given in Eq. (55).

8. Evaluate $e^{[u]}$, where the latent roots of $[u]$ are $\alpha \pm \beta i$.

3·19. High Power of a Matrix. An approximate value of a matrix raised to a high power is easily obtained from Eq. (55). By Eq. (55)

$$[a]^m = \sum_{r=1}^{n} (\lambda_r)^m \frac{\prod_{s\neq r} (\lambda_s I - [a])}{\prod_{s\neq r} (\lambda_s - \lambda_r)}.$$

Let the latent roots of $[a]$ be $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. If m is very large

$$[a]^m \doteq (\lambda_1)^m \frac{\prod_{s\neq 1} (\lambda_s I - [a])}{\prod_{s\neq 1} (\lambda_s - \lambda_1)}. \qquad [56]$$

EXAMPLE. Find an approximate value of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{238}$. By Eq. (56)

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{238} \doteq 2^{238}\frac{(I-[a])}{-1} = -2^{238}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 2^{238}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

[16] J. B. Scarborough, *Numerical Mathematical Methods*, p. 72.

3·20. Matrizant. Tl solution of systems of simultaneous linear differential equations is based on a function called the **matrizant.** The matrizant is defined by the equation

$$\Omega^{t_0t}[u] \equiv I + Q[u] + Q^2[u] + Q^3[u] + \cdots,$$

where $Q = \int_{t_0}^{t} (\ )dt$, $Q^2 = \int_{t_0}^{t} \{(\ ) \int_{t_0}^{t} (\ )dt\}dt$, $\cdots$. Useful properties of the matrizant are displayed in the following theorems.

Theorem I: $\Omega^{t_0t_0}[u] = I$. Proof is mere inspection of the definition.

Theorem II: $\dfrac{d}{dt}\,\Omega^{t_0t}[u] = [u]\,\Omega^{t_0t}[u]$. This result is evident by differentiation of the defining equation.

Theorem III: $\Omega^{t_0t}[a] = I\,e^{[a](t-t_0)}$, where $[a]$ is a constant matrix. This result is established directly from the definition, i.e.,

$$\begin{aligned}\Omega^{t_0t}[a] &= I + Q[a] + Q^2[a] + \cdots \\ &= I + I[a](t - t_0) + I[a]^2(t - t_0)^2/2 + \cdots = I\,e^{[a](t-t_0)}.\end{aligned}$$

3·21. Solution by Matrices. The system of differential equations is first reduced to the normal form of §3·1. The general method of solving simultaneous systems is easily understood from the solution of a simple system. Let it be required to solve the system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2, \\ x_2' &= a_{21}x_1 + a_{22}x_2, \end{aligned} \qquad \text{or} \qquad \frac{d}{dt}[x] = [a][x],$$

where the initial conditions are $x_1(t_0) = x_1^0$ and $x_2(t_0) = x_2^0$. From the method of Sec. 3 and the definition of the matrizant the solution of

$$\frac{d}{dt}\,[x] = [a][x]$$

evidently is

$$[x] = \Omega^{t_0t}[a][x^0],$$

and by theorem III of §3·20,

$$[x] = I\,e^{[a]T}[x^0], \text{ where } T = t - t_0.$$

It remains to compute $e^{[a]T}$. The characteristic equation and latent roots of $[a]$ are respectively

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = 0,$$

$$\lambda_1 = \tfrac{1}{2}\left\{(a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right\} = \alpha + \beta,$$

$$\lambda_2 = \tfrac{1}{2}\left\{(a_{11} + a_{22}) - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right\} = \alpha - \beta.$$

The values of $[Z_1]$ and $[Z_2]$ (see Eq. 55) f $e^{[a]}$ are

$$[Z_1] = -\left\{\frac{(\alpha - \beta)I - [a]}{2\beta}\right\}, \quad [Z_2] = \left\{\frac{(\alpha + \beta)I - [a]}{2\beta}\right\}$$

and

$$e^{[a]T} = \frac{e^{\alpha T}}{2\beta}\{e^{\beta T}([a] - (\alpha - \beta)I) - e^{-\beta T}([a] - (\alpha + \beta)I)\}.$$

Finally, the solution is

$$[x] = I\, e^{\alpha T}\left\{I \cosh \beta T + \frac{1}{\beta}([a] - \alpha I) \sinh \beta T\right\}[x^0]$$

or, in non-matrix notation,

$$x_1 = e^{\alpha T}\left\{x_1^0 \cosh \beta T + \left[\frac{1}{\beta}(a_{11}x_1^0 + a_{12}x_2^0) - \alpha x_1^0\right] \sin \beta T\right\} = 0,$$

$$x_2 = e^{\alpha T}\left\{x_2^0 \cosh \beta T + \left[\frac{1}{\beta}(a_{21}x_1^0 + a_{22}x_2^0) - \alpha x_2^0\right] \sin \beta T\right\} = 0.$$

EXERCISES X

1. Solve, by the matrix method, the system $x_1' = 2x_1$, $x_2' = x_1 + x_2$ subject to the initial conditions $x_i(0) = a_i$.

2. Solve, by the matrix method, the system $x_1' = -x_1 - 2x_2$, $x_2' = x_1 - x_2$ subject to the initial conditions $x_i(0) = a_i$.

3. Solve, by the matrix method, the equation

$$\frac{d^n x}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + a_2\frac{d^{n-2}x}{dt^{n-2}} + \cdots + a_n x = 0,$$

subject to the initial conditions $x(0) = x^0$, $x'(0) = x_1^0$, $\cdots$, $x^{(n-1)}(0) = x_{(n-1)}^0$.

3·22. Vibrations by Means of Matrices. A good approximation to the frequency of the fundamental mode of vibration of a conservative dynamical system with n degrees of freedom can be found by Rayleigh's principle (see §1·39). It is possible to obtain an equally good approximation by means of matrices.

The potential and kinetic energies of a discrete dynamical system are given by Eqs. (54–55) Chap. I.[17] Lagrange's equations (see §1·12) for such a system are

$$\sum_{s=1}^{n} b_{rs}q_s = -\sum_{s=1}^{n} a_{rs}\ddot{q}_s \quad (r = 1, 2, \cdots, n)$$

or

$$[b][q] = -[a][\ddot{q}],$$

[17] Or, see E. T. Whittaker, *Analytical Dynamics*, Chap. VII.

which reduces to

$$[q] = -[D][\ddot{q}], \qquad [57]$$

where $[D] = [b]^{-1}[a]$.

If $q_s = x_s \cos \omega t$ (see §1·37) is substituted in Eqs. (57) we obtain

$$[x] = \omega^2[D][x], \quad \text{or} \quad \frac{1}{\omega^2}[x] = [D][x]. \qquad [58]$$

The determinant $\Delta\left(\frac{1}{\omega^2}\right)$ of the system (58), homogeneous in $x_1, x_2, \cdots, x_n$ is

$$\Delta\left(\frac{1}{\omega^2}\right) = \begin{vmatrix} \frac{1}{\omega^2} - D_{11} & \cdots & \frac{1}{\omega^2} - D_{1n} \\ \cdot & \cdot & \cdot \\ \frac{1}{\omega^2} - D_{n1} & \cdots & \frac{1}{\omega^2} - D_{nn} \end{vmatrix}. \qquad [59]$$

The determinant is also the characteristic determinant of the system of differential equations (57). If, in Eqs. (59), $1/\omega^2$ is replaced by λ then the resulting $\Delta(\lambda)$ is the characteristic determinant of the matrix $[D]$, and $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ are the latent roots of $[D]$. Evidently $1/\sqrt{\lambda_1} = \omega_1$, where ω_1 is the smallest root of Eq. (59) and $\omega_1/2\pi$ is the frequency of the fundamental mode of vibration. (See §1·36.)

It remains only to obtain a simple method of finding λ_1. It is possible to obtain a close approximation to λ_1 from the formula for a high power of a matrix. Equation (56), §3·19, becomes

$$[D]^m \doteq \lambda_1^m \frac{\prod\limits_{s\neq 1} (\lambda_s I - [D])}{\prod\limits_{s\neq 1} (\lambda_s - \lambda_1)},$$

or

$$[D]^m[x^0] \doteq \lambda_1^m \frac{(\lambda_2 I - [D]) \cdots (\lambda_n I - [D])}{(\lambda_2 - \lambda_1) \cdots (\lambda_n - \lambda_1)} [x^0],$$

where the significance of x^0 is given later. Obviously,

$$[D]^{m+1}[x^0] \doteq \lambda_1^{m+1} \frac{(\lambda_1 I - [D]) \cdots (\lambda_n I - [D])}{(\lambda_2 - \lambda_1) \cdots (\lambda_n - \lambda_1)} [x^0].$$

Dividing the last equation by its predecessor we obtain

$$\lambda_1 \doteq \frac{[D]^{m+1}[x^0]}{[D]^m[x^0]}. \qquad [60]$$

The elements of the matrix $[x^0]$ are the values of the coordinates in the estimated fundamental mode. Equation (60), with $[x^0]$ deleted, will ultimately give the value of λ_1. However, reasonable values of $[x^0]$ decrease the value of m which must be employed in Eq. (60). In evaluating $[D]^m$ formula (56) is *not* used, but the value of $[D]^m$ is obtained by *m multiplications*.

EXAMPLE. Obtain the period of the fundamental mode of vibration of the double pendulum of Ex. 3 (see §1·10), where $a = b = 10$ ft., $m_1 = 1$ slug, $m_2 = 2$ slugs, and θ_1 and θ_2 are small.

The differential equations are

$$a(m_1 + m_2)\theta_1'' + bm_2\theta_2'' + (m_1 + m_2)g\,\theta_1 = 0,$$

$$a\theta_1'' + b\theta_2'' + g\theta_2 = 0,$$

which become, on substituting numerical quantities,

$$30\theta_1'' + 20\theta_2'' + 96.6\,\theta_1 = 0,$$

$$10\theta_1'' + 10\theta_2'' + 32.2\,\theta_2 = 0,$$

or

$$[\theta] = -\begin{bmatrix} \dfrac{30}{96.6} & \dfrac{20}{96.6} \\[2ex] \dfrac{10}{32.2} & \dfrac{10}{32.2} \end{bmatrix} [\theta''].$$

Let us estimate that the displacements in the fundamental mode are $x_2^0 = 10 \sin 10° = 1.73$ and $x_2^0 = 20 \sin 10° = 3.46$. We then have

$$[D][x^0] = \begin{bmatrix} \dfrac{30}{96.6} & \dfrac{20}{96.6} \\[2ex] \dfrac{10}{32.2} & \dfrac{10}{32.2} \end{bmatrix} [x^0] = \begin{bmatrix} 0.3105 & 0.2070 \\[2ex] 0.3105 & 0.3105 \end{bmatrix} \begin{bmatrix} 1.73 \\[2ex] 3.46 \end{bmatrix} = \begin{bmatrix} 1.253 \\[2ex] 1.611 \end{bmatrix},$$

$$[D]^2[x^0] = \begin{bmatrix} 0.3105 & 0.2070 \\ 0.3105 & 0.3105 \end{bmatrix} \begin{bmatrix} 1.253 \\ 1.611 \end{bmatrix} = \begin{bmatrix} 0.7229 \\ 0.8903 \end{bmatrix},$$

$$[D]^3[x^0] = \begin{bmatrix} 0.3105 & 0.2070 \\ 0.3105 & 0.3105 \end{bmatrix} \begin{bmatrix} 0.7229 \\ 0.8903 \end{bmatrix} = \begin{bmatrix} 0.4086 \\ 0.5004 \end{bmatrix}.$$

From $[D]^2[x^0]$ and $[D]^3[x^0]$

$$\lambda_1 = \frac{[D]^3[x^0]}{[D]^2[x^0]} = 0.564.$$

We shall carry the process an additional step.

$$[D]^4[x^0] = \begin{bmatrix} 0.3105 & 0.2070 \\ 0.3105 & 0.3105 \end{bmatrix} \begin{bmatrix} 0.4086 \\ 0.5004 \end{bmatrix} = \begin{bmatrix} 0.2305 \\ 0.2822 \end{bmatrix}.$$

From $[D]^3[x^0]$ and $[D]^4[x^0]$

$$\lambda_1 \doteq \frac{[D]^4[x^0]}{[D]^3[x^0]} = 0.564.$$

Evidently, the process has been carried sufficiently far. The approximate value of $\omega_1 = 1/\sqrt{\lambda_1} = 1.332$. The accurate value of ω_1 is 1.34.

EXERCISES XI

1. Three equal weights each of mass m are attached to a light elastic string which is then under tension S. In equilibrium position the length of the string is $4a$ and the three weights are respectively a, $2a$, and $3a$ units from one end of the string. If the coordinates of the three masses are q_1, q_2, q_3, which denote the perpendicular displacements of the three masses from equilibrium position, then the kinetic and potential energies are

$$T = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2),$$

$$V = \frac{S}{2a}[q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2 + q_3^2].$$

Find by the method of §3·22 the period of the fundamental mode of vibration.

2. Two heavy discs, whose moments of inertia are $I_1 = 4$ slug-ft.^{2} and $I_2 = 6$ slug-ft.^{2} are supported on a vertical shaft which is attached to a horizontal plane. The constants of the shaft from the horizontal plane to the first disc and of the shaft between the two discs are respectively $k_1 = 1$ lb. ft./radian and $k_2 = 2$ lb. ft./radian. The energies are

$$T = \tfrac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2), \quad V = \tfrac{1}{2}[k_1\theta_1^2 + k_2(\theta_2 - \theta_1)^2].$$

Find by the method of §3·22 the period of the fundamental mode of vibration.

3. Two pendula, formed by equal masses m and by two rods attached to a horizontal plane, execute vibrations. The two rods are connected by a spring which is attached to the rods a distance h below the two points of support of the rods. The spring constant is k. The length of each bar is l.

Neglecting the weight of each rod and of the spring, calculate, by the method of §3·22, the period of the fundamental mode of vibration of the system.

3·23. Solution by Matrices of Linear Equations with Coefficients Which Are Functions of the Time. From §3·14 it is evident that the solution of the linear system of differential equations

$$\begin{aligned} x_1' &= u_{11}x_1 + \cdots + u_{1n}x_n, \\ & \cdots\cdots\cdots\cdots \\ x_n' &= u_{n1}x_1 + \cdots + u_{nn}x_n, \end{aligned} \quad \text{or} \quad [x'] = [u][x],$$

where $x_i(t_0) = x_i^0$, is given by

$$\Omega^{t_0 t}[u] = I + Qu + Q^2[u] + Q^3[u] + \cdots. \qquad [61]$$

The cumbersomeness, in general, of this formula has been pointed out in §3·15. However, Eq. (61) can be modified as a useful method of numerical integration greatly superior to the method of §65, Vol. I, provided the system of equations is linear.

Let the interval $t_0 \leqq t \leqq t_n$ be divided into n lengths $t_s - t_{s-1} = h_s (s = 1, 2, \cdots, n)$. For simplicity all lengths will be taken equal. Over each interval h_s we shall suppose the matrix $[u]$ to be a constant matrix $[a_s]$ or $[a]$, the elements a_{ij} of which are the average values with respect to t of u_{ij} over the interval h_s whose right end point is s.

The initial conditions for the differential equations at the beginning of the first interval h_1 are $x_i(t_0) = x_i^0$. At the beginning of the sth interval they will be x_i^{s-1}, these values being computed by integrating over the interval whose right end point is $s - 1$. Over the sth interval, since u is assumed constant, Eq. (61) reduces, in view of §3·20 to

$$\Omega^{t_{s-1} t_s}[u] = Ie^{[a_s]T}, \qquad [62]$$

where $[a_s]$ is a constant and $T = t_s - t_{s-1}$.

EXAMPLE. Integrate, by the method of this article, Legendre's equation

$$\frac{d^2x}{dt^2} - \frac{2t}{1 - t^2}\frac{dx}{dt} + \frac{m(m+1)}{1 - t^2}x = 0,$$

subject to the initial conditions $x(0) = -\tfrac{1}{2}$, $x'(0) = 0$. Let $m = 2$.

If $x = x_1$ and $x_1' = x_2$ the normal form of the equation is

$$x_1' = x_2,$$

$$x_2' = -\frac{6x_1}{1-t^2} + \frac{2t}{1-t^2}x_2, \quad \text{or} \quad [x'] = \begin{bmatrix} 0 & 1 \\ \dfrac{-6}{1-t^2} & \dfrac{2t}{1-t^2} \end{bmatrix}[x] = [u][x]. \qquad [63]$$

For the interval $0 \leqq t \leqq 0.1$, Eq. (63) is replaced by

$$[x'] = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}[x] = [a_1][x], \qquad [64]$$

where

$$a_{21} = \frac{-6}{0.1}\int_0^{0.1}\frac{dt}{1 - t^2} = -5.95,$$

$$a_{22} = \frac{2}{0.1}\int_0^{0.1}\frac{t\,dt}{1 - t^2} = 0.1, \quad \text{and} \quad [a_1] = \begin{bmatrix} 0 & 1 \\ -5.95 & 0.1 \end{bmatrix}.$$

The solution of Eq. (64) is

$$[x] = e^{[a]T}x^{0}, \qquad [65]$$

where $x_1^0 = -\frac{1}{2}$ and $x_2^0 = 0$.

The latent roots of $[a_1]$ are complex. The value of $e^{[a]T}$ where the latent roots of $[a]$ are $\alpha \pm \beta i$ is found, by the theory of §3·21 to be

$$e^{[a]T} = \frac{e^{\alpha T}}{\beta}\{(\beta \cos \beta T - \sin \beta T)I + \sin \beta T[a]\}.$$

The values of x_1 and x_2 at $t = 0.1$ (i.e., x_1^1 and x_2^1) are

$$\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \frac{e^{\alpha T}}{\beta}\{(\beta \cos \beta T - \alpha \sin \beta T)I + \sin \beta T[a_1]\}\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix},$$

where $T = 0.1$, $\alpha \pm \beta i = 0.05 \pm 5.95i$, $x_1^0 = -\frac{1}{2}$, $x_2^0 = 0$,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad [a_1] = \begin{bmatrix} 0 & 1 \\ -5.95 & 0.1 \end{bmatrix}.$$

Numerical substitution yields $x_1^1 = -0.42$ and $x_2^1 = 0.28$.

For the interval $0.1 \leqq t \leqq 0.2$ Eq. (63) is replaced by

$$[x'] = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}[x] = [a_2][x],$$

$$a_{21} = \frac{-6}{0.1}\int_{0.1}^{0.2} \frac{dt}{1 - t^2} = -7.5,$$

$$a_{22} = \frac{2}{0.1}\int_{0.1}^{0.2} \frac{t\,dt}{1 - t^2} = 1.17,$$

$$[a_2] = \begin{bmatrix} 0 & 1 \\ -7.5 & 1.17 \end{bmatrix}.$$

The values of x_1^2 and x_2^2 are

$$\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \frac{e^{\alpha T}}{\beta}\{(\beta \cos \beta T - \alpha \sin \beta T)I + \sin \beta T[a_2]\}\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix},$$

where $T = 0.1$, $\alpha \pm \beta i = 0.15 + 4.91i$, $x_1^1 = -0.42$, $x_2^1 = 0.28$. The numerical values are $x_1^2 = -0.37$ and $x_2^2 = 0.49$.

Continuing the process we complete the table of values

t	0.000	0.10	0.20	0.30	0.40	0.50
x_1	−0.500	−0.42	−0.37	−0.31	−0.23	−0.13
x_2	0.000	0.28	0.49	0.73	0.99	1.27

The curves in Fig. 3.5 show both the approximate numerical and also the exact solution $P_2 = 3(t^2 - \frac{1}{3})/2$ over the interval $0 \leqq t \leqq 0.5$. The approximating solution would have been closer to the exact solution if $h = 0.05$ instead of 0.1.

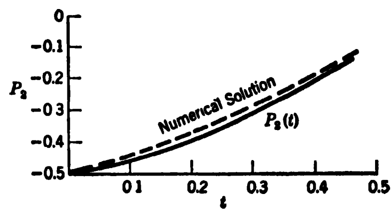

FIG. 3·5

This naturally raises the question as to the magnitude of h if no exact solution is known and, of course, in general in practical problems no exact solution is known. If two numerical solutions are carried out and in one of these the interval h is half its value in the other and if in addition no appreciable difference exists between the two resulting solutions then h is sufficiently small.

The interval h need not be constant throughout the range of the solution. If, in some regions, it is evident that the dependent variable is changing very rapidly as t increases it may be necessary to reduce the value assigned h until a region is reached in which the solution changes more slowly.

It has been emphasized previously that recourse to numerical integration is a last resort. The answers so obtained are merely curves and the parameters of the problem are lost from the solution. If the system contains many parameters and the system is integrated for a series of values of each parameter, either by the method of this section or by means of a mechanical or electrical differential analyzer [18] the solutions will be a book of curves. To express the data thus obtained it is usually necessary to integrate the system of differential equations in an analytical solution.

The most common systems of differential equations whose coefficients are functions of the independent variable and which arise in engineering are those whose coefficients are periodic. (See Ex. 3.) Such equations, even when very simple, may present most formidable difficulties. For certain analytical methods of treating equations of this type see Ref. 14, § 3·47.

[18] See Sec. 10.

EXERCISES XII

1. Solve the illustrative example of § 3·23 employing $h = 0.05$.
2. Integrate, by the method of this article, the differential equation

$$\frac{d^2x}{dt^2} + (16\pi^2e^{-2t} - \tfrac{1}{4})x = 0,$$

subject to the initial conditions $x(0) = 1$, $\dot{x}(0) = 0.5$. Take the range of t to be $0 \leqq t \leqq 2$. Let $h = 0.2$. It is easily verified by substitution that $x = e^{t/2}\cos(4\pi e^{-t})$ is the exact solution for the boundary conditions imposed. Use this solution as a check on the accuracy of the matrix method.

3. Mathieu's equation

$$\frac{d^2u}{dt^2} + (a + 16q\cos 2t)u = 0$$

is of use in two-dimensional wave motion, vibrations of elliptical membranes, astronomy, and free vibratory motion in which there occurs either variable moment of inertia or periodic spring stiffness. The equation possesses periodic or non-periodic solutions dependent upon the values of a and q.

Integrate, by the method of §3·23, Mathieu's equation where $q = 0.1$ and $a = 1 + 8q - 8q^2 - 8q^3 - \frac{8}{3}q^4 +$ insignificant higher degree terms in q. Let the initial conditions and interval of integration be respectively $u(0) = 0$, $u'(0) = 0.5$ and $0 \leqq t \leqq 2\pi$.

It may be advantageous to change the independent variable in the differential equation from t to τ by the relation $2t = \tau$.

4. The differential equation

$$\ddot{x} + 2m\dot{x} + (k^2 - 2n\sin 2t)x = 0$$

is the equation of the free vibrations of a system possessing one degree of freedom, variable spring stiffness, and damping proportional to the first power of the velocity. Integrate, by the method of §3·23, the above equation for $m = 1$ and $n = 0.1$. Let the initial conditions and interval of integration be respectively $x(0) = 0$, $x'(0) = 0.5$ and $0 \leqq t \leqq 2\pi$.

PROBLEM XIII

The matrix method of §3·23 is applicable to linear differential equations only. Originate a matrix method which is valid for systems of non-linear differential equations.

(5)

Elliptic Functions

Elliptic and hyperelliptic functions are of increasing importance in engineering investigations. Problems involving non-linear forces and oscillations whose periods are functions of the initial conditions lead to elliptic functions. Integrals whose integrands contain the square root

of a polynomial of the third or fourth power of the variable of integration are reducible to elliptic integrals. A few of the many elementary applications of elliptic integrals are the length of an ellipse or hyperbola, area of a right elliptic cone, determination of the field intensity at a general point within a circular loop of wire carrying a current,[19] equation of the elastica,[20] equation of a jumping rope,[21] path of a particle moving subject to a central force which is proportional to the inverse fifth power of the distance.[22]

Theories of non-linear springs, non-linear circuits, advanced Schwarzian transformations, and synchronous machines employ elliptic and hyperelliptic functions. (See §§3·31 and 3·35.) An introduction to elliptic functions is necessary for the study of hyperelliptic functions.

3·24. Introductory Problem. Elliptic functions are introduced by the study of the simple pendulum. Let m be the mass of the spherical bob, h the pendulum's length measured from O, the point of suspension, to the center of gravity of m, and θ (Fig. 3·6) the angular displacement of the pendulum at time t. If damping is neglected, the differential equation of motion of the pendulum is

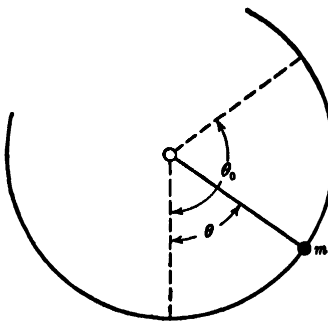

FIG. 3·6

$$\ddot{\theta} + \alpha^2 \sin \theta = 0, \qquad [66]$$

where $\alpha^2 = g/h$ and $g =$ the acceleration of gravity. Integration of (66), after first multiplying the equation through by $2\theta'$, yields

$$\theta'^2 = 2\alpha^2(\cos \theta - \cos \theta_0),$$

where the constant of integration has been so chosen that $\theta' = 0$ for $\theta = \theta_0$. The maximum angular displacement θ_0 is supposed less than π. By the identity $\cos \theta = 1 - 2 \sin^2 \theta/2$ the last equation can be written

$$\theta' = \pm 2\alpha \sqrt{\sin^2 \theta_0/2 - \sin^2 \theta/2}.$$

[19] I. S. and E. S. Sokolnikoff, *Higher Mathematics for Engineers and Physicists*, p. 13.

[20] W. D. MacMillan, *Theoretical Mechanics*, p. 195.

[21] E. B. Wilson, *Advanced Calculus*, p. 511.

[22] W. D. MacMillan, *Theoretical Mechanics*, p. 297.

Change of dependent variable from θ to φ in the above equation, by means of the relation $\sin \theta/2 = (\sin \theta_0/2) \sin \varphi = k \sin \varphi$, yields

$$\varphi' = \alpha \sqrt{1 - k^2 \sin^2 \varphi},$$

or

$$dt = \frac{d\varphi}{\alpha \sqrt{1 - k^2 \sin^2 \varphi}}.$$

If $t = t_0$ when the pendulum is at its low point, integration of the last equation gives

$$\alpha(t - t_0) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.$$

The integral, which is the right member of the last equation, is called an **elliptic integral** of the **first kind**. It cannot be evaluated in terms of a finite number of elementary functions.

3·25. Definitions and Derivatives of the Jacobi Elliptic Functions of a Real Variable. The above equation expresses $t - t_0$ as a function of φ. It is desirable to express φ as an explicit function of $\alpha(t - t_0)$. In so doing we are led to the definitions of elliptic functions. For simplicity in writing, denote $\alpha(t - t_0)$ by u. In the equation

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \qquad [67]$$

the upper limit φ is defined to be the amplitude of u, or in symbols $\varphi = am\, u$. The elliptic functions, sine amplitude, cosine amplitude, and delta amplitude of u, are denoted respectively by $sn\, u$, $cn\, u$, and $dn\, u$ and are defined by the equations

$$\begin{aligned} sn\, u &\equiv \sin am\, u \equiv \sin \varphi, \\ cn\, u &\equiv \cos am\, u \equiv \cos \varphi, \\ dn\, u &\equiv \Delta\, am\, u \equiv \Delta\varphi \equiv \sqrt{1 - k^2 \sin^2 \varphi} = \sqrt{1 - k^2\, sn^2 u}. \end{aligned} \qquad [68]$$

It may be pointed out that the definitions of $sn\, u$ and $cn\, u$ are very similar to the definitions of $\sin u$ and $\cos u$ if the latter definitions are expressed in terms of an integral. That is, if

$$u = + \int_0^{x} \frac{dx}{\sqrt{1 - x^2}}$$

then $u = \sin^{-1} x$ or $u = \cos^{-1} x$ and, consequently, $x = \sin u$ or $x = \cos u$.

The derivatives of *sn u*, *cn u*, and *dn u* are easily obtained. Evidently,

$$\frac{d}{du}\,sn\,u = \frac{d}{du}\sin\varphi = \cos\varphi\frac{d\varphi}{du} = cn\,u\frac{d\varphi}{du}.$$

The value of $\dfrac{d\varphi}{du} = \sqrt{1 - k^2\sin^2\varphi} = dn\,u$ is obtained by differentiating Eq. (67).

Finally,

$$\frac{d}{du}\,sn\,u = cn\,u\,dn\,u.$$

In a similar manner

$$\frac{d}{du}\,cn\,u = -\,sn\,u\,dn\,u, \qquad [69]$$

and

$$\frac{d}{du}\,dn\,u = -\,k^2\,sn\,u\,cn\,u.$$

3·26. Elementary Properties of Elliptic Functions of a Real Variable. It is evident, from Eq. (67), that $am\ 0 = 0$, and consequently $sn\ 0 = 0$, $cn\ 0 = 1$, and $dn\ 0 = 1$. If in Eq. (67) φ is replaced by $-\varphi$ then u changes sign. Thus $am\,(-u) = -am\,u$, and from this fact and the definition of *sn u*, *cn u*, and *dn u* it follows that

$$sn(-u) = -sn\,u,\quad cn(-u) = cn\,u,\quad dn(-u) = dn\,u.$$

The functions *am u* and *sn u* are odd functions; *cn u* and *dn u* are even functions.

The introductory problem of §3·24 can now be completed. From the equation $\sin\theta/2 = k\sin\varphi$ and Eqs. (68)

$$\theta = 2\sin^{-1}(k\,sn\,u) = 2\sin^{-1}[k\,sn\,\alpha(t - t_0)]. \qquad [70]$$

If damping is neglected in the pendulum's motion then the motion will be purely periodic. Equation (66) contains no damping term and consequently θ as given by Eq. (70) is purely periodic. We can study the periodicity of elliptic functions in obtaining the period of the pendulum.

When the pendulum is at its highest point $\varphi = \pi/2$ and the quarter period is

$$\frac{P}{4} = \frac{1}{\alpha}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}.$$

If $k < 1$ then

$$\int_0^{\pi/2} \frac{d\varphi}{\Delta\varphi} = \int_0^{\pi/2} \left[1 + \frac{1}{2} k^2 \sin^2 \varphi + \frac{1\cdot 3}{2\cdot 4} k^4 \sin^4 \varphi + \cdots \right.$$
$$\left. + \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)} k^{2n} \sin^{2n} \varphi + \cdots \right] d\varphi$$
$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots \right.$$
$$\left. + \left(\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)}\right)^2 k^{2n} + \cdots \right].$$

The value of the above integral is denoted by K. Thus the period of the pendulum is $4K(h/g)^{1/2}$. If k is very small, then an approximate value of K is $\pi/2$.

To obtain the real periods of $sn\, u$, $cn\, u$, $dn\, u$, it is necessary to examine the integral

$$\int_0^{n\pi/2} \frac{d\varphi}{\Delta\varphi} = \int_0^{\pi/2} \frac{d\varphi}{\Delta\varphi} + \int_{\pi/2}^{\pi} + \cdots + \int_{(n-1)\pi/2}^{n\pi/2} \frac{d\varphi}{\Delta\varphi},$$

for n a positive integer. Each integral in the above series is of the form

$$\int_{(m-1/2)\pi}^{m\pi} \frac{d\varphi}{\Delta\varphi} \quad \text{or} \quad \int_{m\pi}^{(m+1/2)\pi} \frac{d\varphi}{\Delta\varphi},$$

where m is a positive integer. If in the first integral $\varphi = m\pi - \theta$ and in the second $\varphi = m\pi + \theta$, then each integral becomes

$$\int_0^{\pi/2} \frac{d\theta}{\Delta\theta} = K.$$

Thus

$$\int_0^{n\pi/2} \frac{d\varphi}{\Delta\varphi} = nK$$

and, from the definition of the amplitude function,

$$am(nK) = \frac{n\pi}{2} = n\, am\, K.$$

Consider next the integral

$$\int_0^{n\pi+\beta} \frac{d\varphi}{\Delta\varphi} = \int_0^{n\pi} \frac{d\varphi}{\Delta\varphi} + \int_{n\pi}^{n\pi+\beta} \frac{d\varphi}{\Delta\varphi} = 2nK + u,$$

where

$$u = \int_{n\pi}^{n\pi+\beta} \frac{d\varphi}{\Delta\varphi} = \int_0^{\beta} \frac{d\theta}{\Delta\theta} \quad \text{and} \quad 0 < \beta < \frac{\pi}{2}.$$

From the definition of the amplitude function

$$am(2nK + u) = n\pi + \beta.$$

But $n\pi + \beta = 2n(\pi/2) + \beta = 2n\, am\, K + am\, u$. Thus the important relation

$$am(2nK + u) = 2n\, am\, K + am\, u \qquad [71]$$

is obtained. Similarly, the examination of the integral

$$\int_0^{n\pi-\beta} \frac{d\varphi}{\Delta\varphi} = \int_0^{n\pi} \frac{d\varphi}{\Delta\varphi} + \int_{n\pi}^{n\pi-\beta} \frac{d\varphi}{\Delta\varphi} = 2nK - u$$

yields the formula

$$am(2nK - u) = 2n\, am\, K - am\, u. \qquad [72]$$

Taking the sine of both sides of Eq. (71) we have

$$\sin\,[am(2nK + u)] = \sin\,[2n\, am\, K + am\, u]$$

or

$$\begin{aligned} sn(2nK + u) &= \sin\,(n\pi + am\, u) \\ &= \sin n\pi \cos(am\, u) + \cos n\pi \sin(am\, u) \\ &= \cos n\pi\, sn\, u. \end{aligned}$$

From the last equation, if $n = 2$,

$$sn(u + 4K) = sn\, u.$$

Thus the period of $sn\, u$ is $4K$.

In a similar manner Eqs. (71–72) give the relations

$$sn(u + 2K) = -sn\, u, \quad sn(u + 4K) = sn\, u,$$

$$cn(u + 2K) = -cn\, u, \quad cn(u + 4K) = cn\, u,$$

$$dn(u + 2K) = \sqrt{1 - k^2 sn^2(u + 2K)} = \sqrt{1 - k^2 sn^2 u} = dn\, u.$$

From the last equation the real period of $dn\, u$ evidently is $2K$.

The values of the elliptic integral of Eq. (67) were tabulated by Legendre for values of k less than unity. A five-place table appears in Pierce's *Short Table of Integrals*. From such a table u is given as a function of φ. To obtain the graph of $sn\, u$ it is necessary only to plot $\sin \varphi$ against u as the independent variable. Figure 3·7 shows the graphs of $sn\, u$, $cn\, u$, and $dn\, u$.

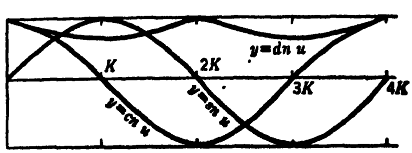

FIG. 3·7

EXERCISES XIV

1. A pendulum beats seconds when swinging through an angle of 6°. How many seconds a day will the pendulum lose if it swings through 10°?

2. The period of a pendulum when swinging through an arc of 72° is two seconds. Find the time required for the same pendulum to swing from 72° down to 52°.

3. In the first integral of the differential equation of a pendulum let the constant of integration be $\frac{2g}{l} + b^2$, where $b^2 > 0$. In this case the angular speed of the pendulum never vanishes. Find the period of revolution.

4. The defining equations for $tn\, u$, $ctn\, u$, $nc\, u$, and $ns\, u$ are

$$tn\, u = \frac{sn\, u}{cn\, u}, \quad ctn\, u = \frac{cn\, u}{sn\, u}, \quad nc\, u = \frac{1}{cn\, u}, \quad ns\, u = \frac{1}{sn\, u}.$$

Obtain the derivatives with respect to u of these four functions.

5. Differentiate

(a) $\log sn\, u$, (d) $-\dfrac{dn\, u}{sn\, u}$,

(b) $\log\left[\dfrac{sn\, u}{cn\, u + dn\, u}\right]$, (e) $-\dfrac{dn\, u}{k'^2\, cn\, u}$, $(k'^2 = 1 - k^2)$,

(c) $\log\left[\dfrac{1 - dn\, u}{sn\, u}\right]$, (f) $\dfrac{1}{(cn\, u - sn\, u)^2}$.

6. The functions $u = sn^{-1}\, x$, $u = cn^{-1}\, x$, $u = tn^{-1}\, x$, and $u = ctn^{-1}\, x$, are defined to be the inverse of $x = sn\, u$, $x = cn\, u$, $x = tn\, u$, and $x = ctn\, u$. Obtain the derivatives with respect to x of the inverse functions.

7. Show that

$$sn^{-1}\, x = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

3·27. Elliptic Integrals. Applications of elliptic functions frequently arise in the form of elliptic integrals.

From the integral calculus it is known that any integral of the type

$$\int R(t, \sqrt{ft^2 + gt + h})\, dt,$$

where R is a rational function of t and of the radical $\sqrt{ft^2 + gt + h}$, is expressible in terms of elementary functions.

It can be shown that integrals of the forms

$$\int R(t, a_1 \sqrt{t^3 + b_1t^2 + c_1t + d_1})\, dt$$

and [73]

$$\int R(t, a \sqrt{t^4 + bt^3 + ct^2 + dt + e})\, dt,$$

where R is a rational function of t and of the radicals can be evaluated in terms of elementary functions and elliptic functions at most. It is supposed that the radicands do not contain multiple factors. The integrals (73) can be reduced to integrals of the elementary calculus and three elliptic integrals:

$$(a)\quad \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

$$(b)\quad \int \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx,$$

$$(c)\quad \int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Legendre's three elliptic integrals, expressed in canonical form, are:
(a) Elliptic integral of the first kind:

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = F(k, x) \quad \text{or} \quad \int_0^\varphi \frac{d\varphi}{\Delta\varphi} = F(k, \varphi),$$

(b) Elliptic integral of the second kind:

$$\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = E(k, x) \quad \text{or} \quad \int_0^\varphi \Delta\varphi\,d\varphi = E(k, \varphi), \quad [74]$$

(c) Elliptic integral of the third kind:

$$\int_0^x \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}} = \Pi(n, k, x) \quad \text{or} \quad \int_0^\varphi \frac{d\varphi}{(1+n\sin^2\varphi)\Delta\varphi},$$

where $\Delta\varphi = \sqrt{1-k^2\sin^2\varphi}$, $0 < k < 1$, and n is a real number.

The coefficients in the polynomials $t^3 + b_1t^2 + c_1t + d_1$ and $t^4 + bt^3 + ct^2 + dt + e$ are real, t is real, and each polynomial is assumed positive for some value of t within the interval of integration.

The second forms of Eqs. (74) are obtained from the first by the change of variable of integration $x = \sin\varphi$. Integrals (74) have been evaluated, by numerical integration and other methods, for all values of k in the interval $0 < k < 1$ and for $0 < \varphi < \pi/2$.

In the introductory problem of §3·24 the elliptic integral was readily reduced to the canonical form of the first kind. This was unusual. In non-linear circuits and Schwarzian transformations the reduction is often tedious. The general reduction is now given.

Let the roots, real or complex, of $t^4 + bt^3 + ct^2 + dt + e$ be α, β, γ, and δ. The real transformation $t = (p + qy)/(1 + y)$ transforms the second integral of Eqs. (73) into

$$\int R_1[y, a\sqrt{Y}](q - p)dy, \qquad [75]$$

where

$$Y = [p - \alpha + (q - \alpha)y][p - \beta + (q - \beta)y]$$
$$[p - \gamma + (q - \gamma)y][(p - \delta) + (q - \delta)y].$$

If the first two factors in Y are multiplied together and the coefficient of the linear term in y set equal to zero there results

$$(p - \alpha)(q - \beta) + (p - \beta)(q - \alpha) = 0. \qquad [76]$$

Treating the last two factors in Y in the same manner we obtain

$$(p - \gamma)(q - \delta) + (p - \delta)(q - \gamma) = 0. \qquad [77]$$

If real values of p and q can be so determined that (76–77) are satisfied then integral (75) will reduce to

$$\int R_0[y, a\sqrt{(\pm m^2 \pm n^2y^2)(\pm r^2 \pm l^2y^2)}](q - p)\, dy,$$

where the real quantities $\pm m^2$, $\pm n^2$, $\pm r^2$, $\pm l^2$ are

$$\pm m^2 = (p - \alpha)(p - \beta), \quad \pm r^2 = (p - \gamma)(p - \delta),$$
$$\pm n^2 = (q - \alpha)(q - \beta), \quad \pm l^2 = (q - \gamma)(q - \delta).$$

(Explicitly, in numerical calculation if $(p - \alpha)(p - \beta) = -7$ then $m^2 = 7$ and the symbols $\pm m^2$ is written -7. If $(p - \alpha)(p - \beta) = 7$, then $\pm m^2$ is written $+7$.) From (76–77)

$$pq + \alpha\beta = \frac{p + q}{2}(\alpha + \beta),$$

$$pq + \gamma\delta = \frac{p + q}{2}(\gamma + \delta).$$

From the last two equations

$$\frac{p + q}{2} = \frac{\alpha\beta - \gamma\delta}{\alpha + \beta - \gamma - \delta}, \quad pq = \frac{\alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta)}{\alpha + \beta - \gamma - \delta}, \qquad [78]$$

which in turn yield

$$\frac{q - p}{2} = \pm\frac{\sqrt{(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)}}{\alpha + \beta - \gamma - \delta}. \qquad [79]$$

From Eqs. (78–79) the real values of p and q are

$$p = \frac{\alpha\beta - \gamma\delta - \sqrt{(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)}}{\alpha + \beta - \gamma - \delta},$$

$$q = \frac{\alpha\beta - \gamma\delta + \sqrt{(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)}}{\alpha + \beta - \gamma - \delta}.$$

The case of most frequent occurrence is

$$R_1(y, a\sqrt{Y}) = \frac{R_0(y)}{\sqrt{Y}},$$

where $R_0(y)$ is a rational function of y. The rational function $R_0(y)$ is the sum of an odd function $R_3(y)$ and an even function $R_2(y)$. Thus the integral is expressed as the sum of two integrals. The integral

$$\int \frac{R_3(y)\,dy}{\sqrt{Y}}$$

is integrable, by means of the substitution $y^2 = u$, by the methods of the calculus. The integration of the integral

$$\int \frac{R_2(y^2)\,dy}{\sqrt{Y}}$$

leads to elliptic integrals. The function $R_2(y^2)$ can be resolved into an integral and a fractional part. The fractional part can be broken up into simple fractions, and by integration by parts, the integration is made to depend upon the terms

$$\frac{dy}{\sqrt{Y}}, \quad \frac{y^2\,dy}{\sqrt{Y}}, \quad \text{and} \quad \frac{dy}{(1 + ny^2)\sqrt{Y}}.$$

We shall carry out in detail the evaluation of the integral

$$\int \frac{dt}{\sqrt{T}} = \int \frac{(q - p)}{\sqrt{Y}}\,dy, \tag{80}$$

where $T = t^4 + at^3 + bt^2 + ct + d$ and the value of Y is given above. The denominator of the integrand in (80) can be written

$$\sqrt{Y} = mr\sqrt{\pm(1 \pm g^2y^2)(1 \pm h^2y^2)},$$

where $g = n/m$ and $h = l/r$. If $g < h$, then the substitution $hy = x$ reduces the right member of (80) to

$$\int \frac{N\,dx}{[\pm(1 \pm x^2)(1 \pm c^2x^2)]^{1/2}}, \tag{81}$$

where $c^2 = (g/h)^2 < 1$ and $N = (q - p)/hmr$.

The eight combinations of sign in the radical of (81) result in eight cases, but the combination of signs $-++$ need not be considered because the polynomial $t^4 + at^3 + bt^2 + ct + d$, which is by hypothesis positive for some range of t within the interval of integration, cannot be transformed by real transformations into a function which is always negative. There exist real transformations which transform the integral in (81) into the integral

$$\frac{1}{M}\int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

where both M and k are real and $0 < k < 1$. The following table indicates the transformation for each combination of sign and gives the corresponding values of k^2 and M.

Sign	Transformation	Value of k^2	Value of M
$+--$	$x = \sin\varphi$	c^2	N
$+-+$	$x = \cos\varphi$	$c^2/(1 + c^2)$	$-N k' = -N(1 - k^2)$
$++-$	$x = (\cos\varphi)/c$	$1/(1 + c^2)$	$-N k$
$+++$	$x = \tan\varphi$	$1 - c^2$	N
$--+$	$x = \sec\varphi$	$1/(1 + c^2)$	$N k$
$-+-$	$x = (\sec\varphi)/c$	$c^2/(1 + c^2)$	$N k'$
$---$	$x^2 = \sin^2\varphi + (\cos^2\varphi)/c^2$	$1 - c^2$	$-N$

If in (74) the upper limit φ is $\pi/2$, then $F(k, \pi/2)$, $E(k, \pi/2)$, and $\Pi(n, k, \pi/2)$ are called **complete elliptic integrals** of the **first**, **second**, and **third kinds** respectively. In the integrals $F(k, x)$, $E(k, x)$, and $\Pi(n, k, x)$ the upper limit may be any real value. Consequently, these integrals may be complex quantities. The explanation of complex values for these integrals is reserved for §3·28.

EXERCISES XV

1. Evaluate by the method of § 3·27 the integrals

$$(a)\quad \int_0^1 \sqrt{1-x^4}\,dx,$$

$$(b)\quad \int_4^5 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)(x-4)}},$$

$$(c)\quad \int_0^2 \frac{dx}{\sqrt{(x^2+x+1)(x^2+x+3)}}.$$

2. If T is of the third degree and if its roots α, β, γ are real and $\alpha > \beta > \gamma$, show that the transformation $t = \gamma + (\beta - \gamma)\sin^2\varphi$ transforms

$$\frac{dt}{\sqrt{T}} \quad \text{into} \quad \frac{2}{\sqrt{\alpha-\gamma}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

where

$$k^2 = \frac{\beta-\gamma}{\alpha-\gamma} \quad \text{and} \quad \gamma < t < \beta.$$

3. (*Reciprocal modulus transformation.*) Show that the transformation $\sin\varphi = (\sin\theta)/c$, where $c > 1$, transforms

$$\frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}} \quad \text{into} \quad \frac{1}{c}\frac{d\theta}{\sqrt{1-\dfrac{1}{c^2}\sin^2\theta}}.$$

4. Plot the integrands of the integrals

$$(a)\quad \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad (b)\quad \int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

for $k = 1/2$. Let φ be taken as abscissa. The areas under the curves give the values of $F(1/2, \varphi)$ and $E(1/2, \varphi)$.

5. Express the integral

$$\int_0^\theta \frac{\sin^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad (0 < k < 1)$$

as the sum of elliptic integrals.

6. Express as elliptic integrals, by proper changes of the variable of integration, the integrals

$$(a)\quad \int_0^\theta \frac{d\theta}{\sqrt{c-\cos\theta}},\ (c > 1), \quad (b)\quad \int_0^{\pi/2} \frac{d\theta}{(\sin\theta)^{1/2}}, \quad (c)\quad \int_0^{\pi/2} \frac{d\theta}{(\cos\theta)^{1/2}}.$$

7. If the four roots of $T = 0$ are $\alpha > \beta > \gamma > \delta$, show that $\dfrac{dt}{\sqrt{T}}$ is transformed into

$$\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

where

$$k^2 = \frac{(\beta - \gamma)(\alpha - \delta)}{(\alpha - \gamma)(\beta - \delta)}, \quad \gamma < t < \beta$$

by the substitution

$$t = \frac{\gamma(\beta - \delta) - \delta(\beta - \gamma)\sin^2\theta}{(\beta - \delta) - (\beta - \gamma)\sin^2\theta}.$$

8. Establish the Maclaurin developments

(a) $$sn\, u = u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} - \cdots$$

(b) $$cn\, u = 1 - \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - (1 + 44k^2 + 16k^4)\frac{u^6}{6!} + \cdots$$

(c) $$dn\, u = 1 - k^2\frac{u^2}{2!} + k^2(4 + k^2)\frac{u^4}{4!} - k^2(16 + 44k^2 + k^4)\frac{u^6}{6!} + \cdots.$$

9. Given that the addition formula for $sn(u + v)$ is $sn(u + v) = \frac{1}{D}(sn\, u\, cn\, v\, dn\, v + sn\, v\, cn\, u\, dn\, u)$, where $D = 1 - k^2\, sn^2 u\, sn^2 v$, show that

$$cn\,(u + v) = \frac{1}{D}(cn\, u\, cn\, v - sn\, u\, dn\, u\, sn\, v\, dn\, v),$$

$$dn\,(u + v) = \frac{1}{D}(dn\, u\, dn\, v - k^2\, sn\, u\, cn\, u\, sn\, v\, cn\, v).$$

3·28. Elliptic Functions of a Complex Variable. Let it be required to examine the integral

$$u = \int_0^{1/k} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}, \qquad [82]$$

where $k < 1$. Evidently

$$u = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} + \int_1^{1/k} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

To transform the last integral write

$$x = \frac{1}{\sqrt{1 - k'^2z^2}},$$

where k' (called the complementary modulus) is defined by the equation $k'^2 + k^2 = 1$. The above transformation changes

$$\int_1^{1/k} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} \quad \text{into} \quad i\int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k'^2z^2)}}$$

or, if $z = \sin\theta$, then into

$$i\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2\theta}} = iK',$$

where K' is given by the series preceding Eq. (71) if k is replaced by k'.

Equation (82) can now be written

$$u = \int_0^{\sin^{-1} 1/k} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = K + iK'.$$

From the definitions of the elliptic functions §3·25

$$am\,(K + iK') = \sin^{-1} 1/k$$

and

$$sn\,(K + iK') = 1/k.$$

In this particular example the sn function of a complex argument yields a real value. We now proceed to the study of elliptic functions of a general argument. In the integral $\int_0^\theta d\theta/\Delta(\theta, k)$ make the substitution

$$\cos\theta\cos\varphi = 1. \qquad [83]$$

Then $\sin\theta = i\tan\varphi$ and

$$\int_0^\theta \frac{d\theta}{\Delta(\theta, k)} = i\int_0^\varphi \frac{d\varphi}{\Delta(\varphi, k')}.$$

If

$$\int_0^\varphi \frac{d\varphi}{\Delta(\varphi, k')} = u \qquad [84]$$

so that

$$\varphi = am(u, k')$$

then

$$\int_0^\theta \frac{d\theta}{\Delta(\theta, k)} = iu$$

and

$$\theta = am(iu, k). \qquad [85]$$

From relation (83) there follows immediately

$$\sin\theta = i\tan\varphi,$$

$$\cos\theta = 1/\cos\varphi,$$

$$\tan\theta = i\sin\varphi.$$

Substituting the values of φ and θ from (84–85) in the last equations we have, from $\sin\theta = i\tan\varphi$, the relation

$$\sin am(iu, k) = i \tan am(u, k'),$$

or

$$sn(iu, k) = i\, tn(u, k').$$

Likewise

$$cn(iu, k) = 1/cn(u, k'), \qquad [86]$$

$$dn(iu, k) = \frac{dn(u, k')}{cn\,(u, k')}.$$

If in (86) u is replaced by $v + 4K'$ then the last equations yield

$$sn[i(v + 4K'), k] = i\, tn\,[(v + 4K'), k'] = i\, tn\,(v, k') = sn(iv, k).$$

$$cn[i(v + 4K'), k] = cn(iv, k), \qquad [87]$$

$$dn[i(v + 4K'), k] = dn(iv, k).$$

If in (87) v is replaced by $-iv$ (and this is a possible substitution by inspection of the definitions of u and v) there results

$$sn(v + 4iK', k) = sn(v, k),$$

$$cn(v + 4iK', k) = cn(v, k), \qquad [88]$$

$$dn(v + 4iK', k) = dn(v, k).$$

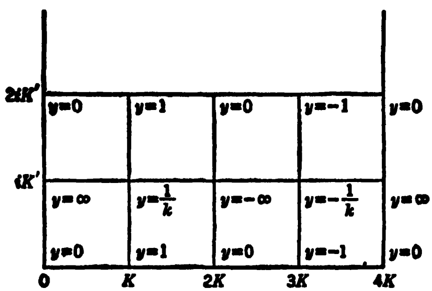

FIG. 3·8. Rectangle of Elliptic Function $sn\ u$.

It can now be shown that the periods of $sn\ u$, $cn\ u$ and $dn\ u$ are respectively ($4K$ and $2iK'$), ($4K$ and $2K + 2iK'$) and ($2K$ and $4iK'$). Thus all values of $y = sn\ u$ are given in the rectangle shown in Fig. 3·8.

EXERCISES XVI

1. Fill in a rectangle, similar to that shown in Fig. 3·8, for the function $y = cn\, u$.
2. Show that

$$sn\left(\frac{K}{2}+\frac{iK'}{2}\right) = \frac{1}{\sqrt{2k}}(\sqrt{1+k}+i\sqrt{1-k}),$$

$$cn\left(\frac{3K}{2}+\frac{iK'}{2}\right) = -\frac{(1+i\sqrt{k'})}{\sqrt{2k}}.$$

3. Show that

$$(a)\quad sn\,\frac{3K}{2} = \frac{1}{\sqrt{1+k'}},\quad (b)\quad cn\,\frac{3K}{2} = -\frac{\sqrt{k'}}{\sqrt{1+k'}},\quad (c)\quad dn\,\frac{3K}{2} = \sqrt{k'}.$$

4. Show that

$$sn\left[(1+k)u, \frac{2\sqrt{k}}{1+k}\right] = \frac{(1+k)\, sn\,(u,k)}{1+k\, sn^2\,(u,k)}.$$

5. Express

$$\int_0^{10} \frac{dx}{\sqrt{(1-x^2)\,(1-(1/2)^2x^2)}}$$

as a complex number $A + Bi$.

3·29. Integration of Elliptic Functions. The methods of evaluation of integrals whose integrands contain elliptic functions are very similar to the methods of the elementary calculus. From

$$u = \int_0^{\varphi} \frac{d\varphi}{\Delta\varphi}$$

it follows that

$$du = d\varphi/\Delta\varphi, \quad \text{or} \quad d\varphi = dn\,u\,du, \quad \text{or} \quad d(am\,u) = dn\,u\,du.$$

From the last equations,

$$d(sn\,u) = cn\,u\,dn\,u\,du,$$

$$d(cn\,u) = -sn\,u\,dn\,u\,du,$$

$$d(dn\,u) = -k^2\,sn\,u\,cn\,u\,du.$$

Some methods of integration are illustrated by the following examples.

1. Evaluate the integral $\int sn\,u\,du$. (Omit the arbitrary constant.) The integral

$$\int sn\,u\,du = -\frac{1}{k^2}\int \frac{-k^2\,sn\,u\,cn\,u}{cn\,u}\,du = -\frac{1}{k}\int \frac{dv}{\sqrt{v^2-k'^2}},$$

where $v = dn\,u$. The value of the last integral is

$$-\frac{1}{k}\log\left(v+\sqrt{v^2-k'^2}\right) = -\frac{1}{k}\cosh^{-1}\frac{v}{k'} = -\frac{1}{k}\cosh^{-1}\left(\frac{dn\,u}{k'}\right).$$

2. The evaluation of $\int du/sn\,u$ is as follows:

$$\int\frac{du}{sn\,u} = \int\frac{sn\,u\;cn\,u\;dn\,u}{sn^2\,u\;cn\,u\;dn\,u}\,du = \frac{1}{2}\int\frac{dv}{v\sqrt{(1-v)(1-k^2v)}},$$

where $v = sn^2\,u$. Evaluating the last integral by the methods of the calculus and expressing the result in terms of u it is found that

$$\int\frac{du}{sn\,u} = \log\left[\frac{cn\,u}{cn\,u + dn\,u}\right].$$

3. Evaluate the integral $\int sn^{-1}\,u\,du$. In the integrand make the substitution $sn^{-1}\,u = v$ or $u = sn\,v$. Then

$$\int sn^{-1}\,u\,du = \int v\;cn\,v\;dn\,v\,dv.$$

Integration by parts yields

$$\int sn^{-1}\,u\,du = \int v\;cn\,v\;dn\,v\,dv = v\;sn\,v + \frac{1}{k}\cosh^{-1}\left(\frac{v}{k'}\right)$$
$$= u\;sn^{-1}\,u + \frac{1}{k}\cosh^{-1}\left(\frac{sn^{-1}\,u}{k'}\right).$$

4. Evaluate the integral $\int_0^u dn^2\,u\,du$. By definition $E(k,\varphi) = \int_0^\varphi \Delta\varphi\,d\varphi$. We have $\varphi = am\,u$ and $d(am\,u) = d\varphi = dn\,u\,du$. Recalling that $\Delta\varphi = dn\,u$, and substituting for $\Delta\varphi$ and $d\varphi$ in the last integral we have

$$\int_0^\varphi dn^2\,u\,du = E(k, am\,u).$$

5. Evaluate the integrals $\int_0^u sn^2\,u\,du$ and $\int_0^u cn^2\,u\,du$. From the definitions of $sn\,u$, $cn\,u$, and $dn\,u$ the relations

$$sn^2\,u + cn^2\,u = 1,$$
$$dn^2\,u + k^2\,sn^2\,u = 1$$

obtain. By means of these relations the evaluation of the two integrals in question is made to depend upon the integral of example 4. The results are

$$\int_0^u sn^2\, u\, du = \frac{1}{k^2}\,[u - E(am\, u, k)],$$

$$\int_0^u cn^2\, u\, du = \frac{1}{k^2}\,[E(am\, u, k) - k'^2\, u].$$

6. Evaluate the integral $\displaystyle\int_0^x \frac{dx}{\sqrt{(1 + x^2)(1 + k^2x^2)}}$.

If the substitution $x = tn(u, k')$ is made then

$$dx = \frac{dn(u, k')\, du}{cn^2(u, k')}, \qquad 1 + k^2\, tn^2(u, k') = \frac{1 - (1 - k^2)sn^2(u, k')}{cn^2(u, k')}$$

$$1 + x^2 = \frac{1}{cn^2(u, k')}, \qquad\qquad = \frac{1 - k'^2\, sn^2(u, k')}{cn^2(u, k')}$$

$$= \frac{dn^2(u, k')}{cn^2(u, k')},$$

and the integral becomes

$$\int_0^x \frac{dx}{\sqrt{(1 + x^2)(1 + k^2x^2)}} = \int \frac{dn(u, k')cn^2(u, k')}{cn^2(u, k')dn(u, k')}\, du$$

$$= \int du = u = tn^{-1}\,(x, k')$$

$$= sn^{-1}\left(\frac{x}{\sqrt{1 + x^2}}, k'\right) = F(k', \tan^{-1} x).$$

A rather extensive table of integrals for elliptic functions is found in *Eléments de la Théorie des Fonctions Elliptiques* IV, Tannery and Molk.

EXERCISES XVII

Establish the following formulas:

1. $\displaystyle\int_0^u \frac{du}{dn^2\, u} = \frac{E(u)}{k'^2} - \frac{k^2\, sn\, u\, cn\, u}{k'^2\, dn\, u}$,

2. $\displaystyle\int_0^u tn\, u\, du = \frac{1}{k'} \log \frac{dn\, u + k'}{cn\, u}$,

3. $\displaystyle\int ctn\, u\, du = \log \frac{1 - dn\, u}{sn\, u},$

4. $\displaystyle\int \frac{sn\, u\, du}{cn^2\, u} = -\frac{1}{k'^2}\frac{dn\, u}{cn\, u},$

5. $\displaystyle\int dn\, u = i \log (cn\, u - i\, sn\, u),$

6. $\displaystyle\int \frac{cn\, u\, dn\, u}{sn\, u}\, du = \log sn\, u,$

7. $\displaystyle\int \frac{dn\, u\, du}{cn^2\, u} = tn\, u,$

8. $\displaystyle\int \frac{du}{sn\, u\, cn\, u} = \int \frac{cn\, u}{sn\, u}\, du + \int \frac{sn\, u}{cn\, u}\, du.$

Evaluate, by substitution of elliptic functions (see §3·29), the integrals

9. $\displaystyle\int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}},$ $(0 < k < 1)$. Let $x = sn\, u$.

10. $\displaystyle\int_x^b \frac{dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)}},$

11. $\displaystyle\int_0^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}},$

12. $\displaystyle\int_0^x \frac{dx}{\sqrt{x(1 - x)(1 - k^2x)}},$

13. $\displaystyle\int \frac{dx}{\sqrt{(x - a)[(x - r)^2 + s^2]}}.$

Let $y = \dfrac{(x - r)^2 + s^2}{x - a}$ and obtain under the radical sign an expression having .hree real factors. Then use Ex. 2, problem set XV.

(6)

Hyperelliptic Functions

Some of the uses of hyperelliptic functions have been given in the introductory paragraph of Sec. 5. It is the purpose of this section to develop the theory, sufficient for the applications considered, of hyperelliptic functions. Integrals of the form

$$\int R(t, \sqrt{T})dt,$$

where T is a polynomial of degree higher than four and R is a rational function of t and $\sqrt{T}$, lead in general to hyperelliptic functions. It is not easy to generalize the classical theory of elliptic functions as given in Sec. 5 so as to obtain a theory of hyperelliptic functions. A theory of elliptic functions is now developed which is based on the solution of differential equations by the methods of Sec. 1 of this chapter. This theory will then be extended so as to include hyperelliptic functions.

3·30. Elliptic Functions in Series Form. Consider the differential equation

$$\frac{d^2x}{dt^2} = -(1 + k^2)x + 2k^2x^3, \quad 0 < k^2 < 1. \tag{89}$$

If Eq. (89) is multiplied through by $2\,dx/dt$ and the integration performed there results

$$x'^2 = (1 - x^2)(1 - k^2 x^2) \tag{90}$$

provided $x'(0) = 1$, and $x(0) = 0$. The solution of Eq. (90), satisfying these initial conditions, is

$$x = sn(t, k).$$

In view of the theory of Sec. 1 it seems reasonable to suppose that the solution of Eq. (89), subject to the initial conditions $x'(0) = 1$ and $x(0) = 0$, is obtainable as a power series in k^2. Accordingly, let

$$x = x_0(t) + x_2(t)k^2 + x_4(t)k^4 + \cdots.$$

Substituting this value of x in Eq. (89) and equating the coefficients of like powers of k^2 we have the sequence of equations

$$\begin{aligned} x_0'' + x_0 &= 0, \\ x_2'' + x_2 &= -x_0 + 2x_0^3, \\ x_4'' + x_4 &= -x_2 + 6x_0^2\, x_2 \\ \cdots & \cdots\cdots \end{aligned} \tag{91}$$

The solution, subject to the initial conditions, of the first of (91) is

$$x_0 = \sin t.$$

The substitution of x_0 in the second equation above yields

$$x_2'' + x_2 = \tfrac{1}{2}\sin t - \tfrac{1}{2}\sin 3t.$$

The solution, subject to the initial conditions $x_2'(0) = x_2(0) = 0$, is

$$x_2 = \frac{1}{16}\sin t - \frac{t}{4}\cos t + \frac{1}{16}\sin 3t.$$

The solution of Eq. (89), as far as the terms in k^2, is

$$x = \sin t + \frac{k^2}{16}(\sin t - 4t\cos t + \sin 3t) + \cdots. \qquad [92]$$

Evidently, the solution (92) is not satisfactory. The term $-(t\cos t)/4$ is not periodic. Moreover, the remaining terms of (92) are periodic of period 2π, whereas the solution $x = sn(t, k)$ is known to be periodic of period $4K$ in t.

A solution must be devised that displays the period of $4K$ in t. Accordingly, let a change of independent variable from t to τ be made in Eq. (89) by the relation $t = (1 + \delta)\tau$, where δ is a constant later determined. Since

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{(1+\delta)}\frac{dx}{d\tau} \quad \text{and} \quad \frac{d^2x}{dt^2} = \frac{1}{(1+\delta)^2}\frac{d^2x}{d\tau^2},$$

Eq. (89) becomes

$$\frac{d^2x}{d\tau^2} = -(1+\delta)^2[(1+k^2)x - 2k^2x^3]. \qquad [93]$$

From § 3·26 the value of $4K$ is

$$2\pi\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right].$$

If the solution of Eq. (93) can be expressed in a form which is periodic of period 2π then by the relation $t = (1 + \delta)\tau$, the solution in t will be periodic of period $2\pi(1 + \delta)$.

Since δ is a function of k^2 it is reasonable to write $\delta = \delta_2k^2 + \delta_4k^4 + \cdots$, where $\delta_2, \delta_4, \cdots$ are constants to be determined. The substitution of this value for δ and $x = x_0(\tau) + x_2(\tau)k^2 + x_4(\tau)k^4 \cdots$ in Eq. (93) gives

$$\frac{d^2x_0}{d\tau^2} + \frac{d^2x_2k^2}{d\tau^2} + \frac{d^2x_4k^4}{d\tau^2} + \cdots = \{-x_0 - k^2[x_2 + (2\delta_2 + 1)x_0 - 2x_0^3] -$$
$$k^4[x_4 + (2\delta_4 + \delta_2^2 + 2\delta_2)x_0 + (2\delta_2 + 1)x_2 - 6x_0^2x_2 - 4\delta_2x_0^3] + \cdots\}.$$

By equating coefficients of like powers of k^2 we obtain the infinite sequence of linear differential equations

$$\begin{aligned} &x_0'' + x_0 = 0, \\ &x_2'' + x_2 = -(1 + 2\delta_2)x_0 + 2x_0^3, \\ &x_4'' + x_4 = -(2\delta_2 + \delta_2^2 + 2\delta_4)x_0 - (1 + 2\delta_2)x_2 + 6x_0^2x_2 + 4\delta_2x_0^3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad [94]$$

where the derivatives are with respect to τ.

The solution of the first of (94), subject to the initial conditions $x_0(0) = 0$, $x_0'(0) = 1$, is $x_0 = \sin \tau$. When $\sin \tau$ is substituted for x_0 in the second equation above, there results

$$x_2'' + x_2 = (\tfrac{1}{2} - 2\delta_2) \sin \tau - \tfrac{1}{2} \sin 3\tau.$$

In order that no term of the form $\tau \cos \tau$ appears in the solution, it is necessary that $\frac{1}{2} - 2\delta_2 = 0$ or $\delta_2 = \frac{1}{4}$.

Before integrating the next differential equation and the remaining equations of (94), it is necessary to determine the initial conditions for $x_2, x_4, \cdots$. From $t = (1 + \delta)\tau$ it follows that

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = (1 + \delta)\frac{dx}{dt} = (1 + \delta_2 k^2 + \delta_4 k^4 + \cdots)\frac{dx}{dt}.$$

From Eq. (90) it is evident that $\dfrac{dx}{dt} = 1$ for $x = 0$, $t = 0$. Thus

$$\frac{dx}{d\tau} = 1 + \delta_2 k^2 + \delta_4 k^4 + \cdots, \text{ for } \tau = 0.$$

From $x = x_0 + x_2(\tau)k^2 + x_4(\tau)k^4 + \cdots$, a second value for $\dfrac{dx}{d\tau}$ is

$$\frac{dx}{d\tau} = \frac{dx_0}{d\tau} + \frac{dx_2}{d\tau}k^2 + \frac{dx_4}{d\tau}k^4 + \cdots.$$

Since these two values are identical for all values of k when $\tau = 0$, it follows that, at $\tau = 0$,

$$\frac{dx_0}{d\tau} = 1, \quad \frac{dx_2}{d\tau} = \delta_2, \cdots, \frac{dx_{2n}}{d\tau} = \delta_{2n}.$$

The solution of $x_2'' + x_2 = -\frac{1}{2} \sin 3\tau$, subject to the initial conditions $x_2(0) = 0$ and $x_2'(0) = \delta_2$ is

$$x_2 = \tfrac{1}{16} (\sin \tau + \sin 3\tau).$$

By substituting x_0 and x_2 in the third equation of (94) and imposing the condition that all $\sin \tau$ terms vanish from the right member of the equation, δ_4 is found to be $\left(\dfrac{1 \cdot 3}{2 \cdot 4}\right)^2$. Moreover, it is evident that in each successive equation just one additional δ_{2n} enters. Thus the δ_{2n} can be determined. The integration of the equation in x_4 gives

$$x_4 = \frac{1}{16^2} (7 \sin \tau + 8 \sin 3\tau + \sin 5\tau).$$

Finally, the value of $sn(t, k)$ as far as the terms containing k^4 is

$$sn(t, k) = x_0 + x_2 k^2 + x_4 k^4 + \cdots, \qquad [95]$$

where

$$x_0 = \sin \tau = \sin \frac{t}{1+\delta} = \sin \frac{\pi t}{2K},$$

$$x_2 = \frac{1}{16}\left(\sin \frac{\pi t}{2K} + \sin \frac{3\pi t}{2K}\right),$$

$$x_4 = \frac{1}{16^2}\left(7 \sin \frac{\pi t}{2K} + 8 \sin \frac{3\pi t}{2K} + \sin \frac{5\pi t}{2K}\right).$$

The solution of (89) given by (95) is periodic of period $4K$ and is of a form satisfactory for computational purposes.

EXERCISES XVIII

1. Solve, by the method of §3·30, the differential equation

$$\frac{d^2x}{dt^2} = -(1+k^2)x + 2k^2x^3, \quad (0 < k^2 < 1)$$

subject to the initial conditions $x(0) = 1$, $x'(0) = 0$ and thus obtain a series expansion, similar in form to (95), for *cn t*.

2. Obtain the differential equation whose solution is *dn t*. Show that the solution, satisfying the initial conditions $x(0) = 1$, $x'(0) = 0$ is

$$x = 1 + x_2k^2 + x_4k^4 + \cdots,$$

$$x_2 = -\frac{1}{2}\sin^2 \frac{\pi t}{2K},$$

$$x_4 = -\frac{1}{8}\sin^2 \frac{\pi t}{2K}\left(1 + \cos^2 \frac{\pi t}{2K}\right),$$

$$\cdots\cdots\cdots\cdots$$

3·31. Non-linear Spring. Let it be required to find the period of oscillation of the mass m supported by a non-linear spring as illustrated in Fig. 3·9. The displacement from equilibrium position at time t is denoted by x. Let the restoring force of the spring be given by

$$F = b_1x + b_3x^3,$$

where b_1 and b_3 are positive empirical constants.

The differential equation of motion is

$$m\frac{d^2x}{dt^2} = -(b_1x + b_3x^3),$$

or

$$\frac{d^2x}{dt^2} = -(a_1x + a_3x^3).$$

The first integral of the differential equation is

$$\left(\frac{dx}{dt}\right)^2 = -\left(a_1x^2 + \frac{a_3x^4}{2}\right) + C_1.$$

Denote the maximum displacement of the spring from equilibrium position by c. Since the velocity vanishes at maximum displacement $C_1 = a_1c^2 + a_3c^4/2$. The differential equation becomes

$$\left(\frac{dx}{dt}\right)^2 = a_1(c^2 - x^2) + \frac{a_3}{2}(c^4 - x^4).$$

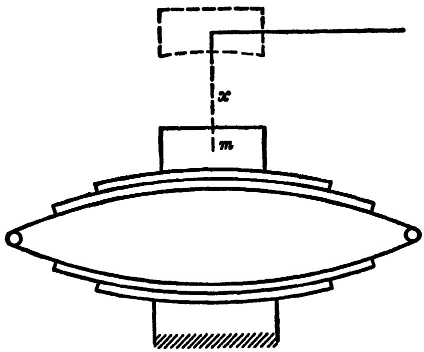

FIG. 3·9. Non-linear Spring.

The substitution $x = cy$ reduces the last equation to

$$\left(\frac{dy}{dt}\right)^2 = \frac{a_3c^2}{2k^2}(1 - y^2)(1 + k^2y^2),$$

where

$$k^2 = \frac{a_3c^2}{2a_1 + a_3c^2} < 1.$$

When y ranges from -1 to $+1$ the mass has executed a half-period $P/2$. Thus

$$\begin{aligned} P &= \frac{2k}{c}\left(\frac{2}{a_3}\right)^{1/2}\int_{-1}^{+1}\frac{dy}{\sqrt{(1 - y^2)(1 + k^2y^2)}} \\ &= \frac{4k}{c}\left(\frac{2}{a_3}\right)^{1/2}\int_0^1\frac{(1 + k^2y^2)^{-1/2}}{\sqrt{1 - y^2}}\,dy \\ &= \frac{4k}{c}\left(\frac{2}{a_3}\right)^{1/2}\int_0^1\left(1 - \frac{1}{2}k^2y^2 + \frac{1\cdot3}{2\cdot4}k^4y^4 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}k^6y^6 + \cdots\right) \\ &\qquad\qquad \frac{dy}{\sqrt{1 - y^2}}. \end{aligned}$$

The trigonometric substitution $y = \sin\theta$ and Wallis' integration formulas yield

$$P = 2\pi\frac{k}{c}\left(\frac{2}{a_3}\right)^{1/2}\left[1 - \frac{k^2}{4} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right]$$

$$= 2\pi\frac{k}{c}\left(\frac{2m}{b_3}\right)^{1/2}\left[1 - \frac{k^2}{4} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right].$$

It should be noted that the period, unlike the period of a linear spring, is a function of the maximum displacement of the mass.

EXERCISES XIX

1. Solve by the method of §3·31 the differential equation

$$\frac{d^2y}{dt^2} = -\frac{a_3}{4k^2}[(1 - k^2)y + 2k^2y^3],$$

subject to the initial conditions $y(0) = 0$, $y'(0) = a_3/4k^2$. Show that the action of the spring is rougher than a linear spring.

3·32. Hyperelliptic Functions. Hyperelliptic functions may be introduced by the equation

$$\frac{d^2z}{dt^2} = b_0 + b_1z + b_2z^2 + \cdots + b_{n-1}z^{n-1}, \qquad [96]$$

where $b_0, b_1, b_2, \cdots, b_{n-1}$ are real constants and the right member is a polynomial of degree greater than four or a convergent infinite series. (Ref. 4, end of chapter.) A first integral of Eq. (96) is

$$\left(\frac{dz}{dt}\right)^2 = 2\left(b_0z + \frac{b_1}{2}z^2 + \cdots + \frac{b_{n-1}}{n}z^n\right) + C$$
$$= a_0 + a_1z + a_2z^2 + \cdots + a_nz^n = f(z), \qquad [97]$$

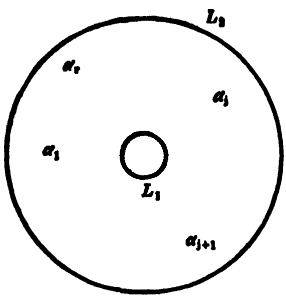

FIG. 3·10

where $a_0, a_1, \cdots, a_n$ are all real. Let the n roots, real or complex, of $f(z)$ be $\alpha_1, \alpha_2, \cdots, \alpha_n$. In any physical problem the variation of z will lie between fixed values Z_1 and Z_2 where Z_1 and Z_2 are real or complex. Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be those roots of $f(z)$ which lie within the ring (Fig. 4·10) $L_1 \leqq z \leqq L_2$. Let $f(z)$ be written

$$f(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\cdots(z - \alpha_r)f_0(z), \qquad [98]$$

where $f_0(z)$ is finite and does not vanish within the ring.

3·33. Period of the Solution. If $n = 4$, it is evident from §3·28 that the periods of the solution of (96) will be one real and one imaginary. Thus in the general equation $n > 4$, both real and imaginary periods are expected, but in the problems of this section, only real periods need be considered.

The real periods in the general case are obtained in much the same way that the period of the non-linear spring was obtained in §3·31. Let us suppose that during the motion or variation of current z varies from α_i to α_{i+1} where z, α_i, and α_{i+1} are real and that $f(z) > 0$ for $\alpha_i \leqq z \leqq \alpha_{i+1}$. If the change of variable

$$z = \frac{(\alpha_i + \alpha_{i+1})}{2} + \frac{(\alpha_{i+1} - \alpha_i)}{2} x$$

is made in $f(z)$, then

$$\begin{aligned}
f(z) = &\left(\frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} x - \alpha_1\right) \times \\
&\left(\frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} x - \alpha_2\right) \cdots \times \\
&\left(\frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} x - \alpha_i\right) \times \\
&\left(\frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} x - \alpha_{i+1}\right) \cdots \times \\
&\left(\frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} x - \alpha_r\right) G(x) \\
= &\frac{(\alpha_{i+1} - \alpha_i)^2}{4} (1 - x^2) \prod_{j=1}^{r} \left[\frac{\alpha_i + \alpha_{i+1} - 2\alpha_j}{2} + \frac{(\alpha_{i+1} - \alpha_i)x}{2}\right] G(x), \\
= &\frac{(\alpha_{i+1} - \alpha_i)^2}{4} (1 - x^2) \prod_{j=1}^{r} (b_j + cx) G(x), \\
= &\frac{(\alpha_{i+1} - \alpha_i)^2}{4} (1 - x^2) \prod_{j=1}^{r} b_j \left(1 + \frac{c}{b_j} x\right) G(x),
\end{aligned}$$

where

$$\frac{c}{b_j} = \left|\frac{\alpha_{i+1} - \alpha_i}{\alpha_{i+1} + \alpha_i - 2\alpha_j}\right| < 1$$

because α_{i+1} and α_i are consecutive roots of $f(x)$. If $b_j\left(1 + \frac{c}{b_j} x\right)$ is a complex root, then its complex conjugate (say) $b_{j+1}\left(1 + \frac{cx}{b_{j+1}}\right)$

is also a root. The coefficients d_j and e_j in the product

$$b_j b_{j+1}\left(1+\frac{cx}{b_j}\right)\left(1+\frac{cx}{b_{j+1}}\right) = B^2(1 + d_j x + e_j x^2)$$

are real numbers and B^2 is a positive quantity. Finally,

$$f(x) = \frac{A^2}{4}(\alpha_{i+1} - \alpha_i)^2(1 - x^2)(1 + \sigma_1 x) \cdots$$
$$(1 + \sigma_p x)(1 + d_1 x + e_1 x^2) \cdots (1 + d_l x + e_l x^2) g(x),$$

where the number of real roots of $f(z)$ in the ring $L_1 < |x| < L_2$ is p, the number of complex roots is $2l$ and A^2 is such that $g(0) = 1$. The change of variable from z to x reduces (97) to

$$dt = \int \frac{dx}{A\sqrt{(1-x^2)(1+\sigma_1 x)\cdots(1+\sigma_p x)\cdots(1+d_1 x+e_1 x^2)\cdots(1+d_l x+e_l x^2)g(x)}}.$$

The period $T(\alpha_{i+1}, \alpha_i)$ is

$$T(\alpha_{i+1}, \alpha_i) = \frac{2}{A}\int_{-1}^{+1} \frac{\left\{\begin{matrix}(1 + \sigma_1 x) \cdots (1+\sigma_p x)(1+d_1 x+e_1 x^2)\\ \cdots (1 + d_l x + e_l x^2) g(x)\end{matrix}\right\}^{-\frac{1}{2}} dx}{\sqrt{1 - x^2}}.$$

Since $g(x)$ does not vanish within the interval of integration $[g(x)]^{-\frac{1}{2}}$ is expansible in this interval as a convergent power series in x. Since $\sigma_1, \sigma_2, \cdots, \sigma_p$ are less than unity $(1 + \sigma_1 x) \cdots (1 + \sigma_p x)$ is also expansible as a power series in x for the interval in question.

Two cases now obtain:

Case (*a*). If $|x| < 1/\sqrt{|e_j|}$, $(j = 1, 2, \cdots, l)$ then each factor $\sqrt{(1 + d_j x + e_j x^2)}$ can be expanded as a power series in x. Thus $T(\alpha_{i+1}, \alpha_i)$ becomes

$$T(\alpha_{i+1}, \alpha_i) = \frac{2}{A}\int_{-1}^{+1} \frac{(1 + D_1 x + D_2 x^2 + \cdots)}{\sqrt{1 - x^2}}\, dx,$$

where $D_1, D_2, \cdots$ are constants. By the trigonometic substitution $x = \sin\theta$ the integral is easily evaluated and the period obtained.

Case (*b*). If any or all of the $|e_j|$ are greater than unity the procedure is more complicated. Consider first that one of the $|e_j| > 1$ and the absolute value of each of the others is less than unity. Let the corresponding factor be written

$$1 + d_j x + e_j x^2 = e_j[x - (\alpha + \beta i)][x - (\alpha - \beta i)],$$

where α and β are real. Let the integral $T(\alpha_{i+1}, \alpha_i)$ be computed from -1 to α and then from α to $+1$. Over the first range make the change of variable

$$x = \frac{\alpha - 1}{2} + \frac{\alpha + 1}{2} y.$$

Then

$$\{e_j[x - (\alpha + \beta i)][x - (\alpha - \beta i)]\}^{-1/2} = \{e_j[(x - \alpha)^2 + \beta^2]\}^{-1/2}$$

$$= \left\{e_j\left[\frac{(1 + \alpha)^2 + 4\beta^2}{4}\right]\left[1 - \frac{2(1 + \alpha)^2 y}{(1 + \alpha)^2 + 4\beta^2} + \frac{(1 + \alpha)^2 y^2}{(1 + \alpha)^2 + 4\beta^2}\right]\right\}^{-1/2}.$$

The last factor is expansible in a convergent power series in y as long as $|y| < \sqrt{1 + \frac{4\beta^2}{(1 + \alpha)^2}}$. Since this inequality is satisfied for the interval $-1 < y < \alpha$

$$T(\alpha_i, \alpha) = \int_{-1}^{+1} \frac{(Q + Qy + Qy^2 + \cdots)}{\sqrt{1 - y}} dy,$$

where the Q_i are constants.

For the interval $\alpha \leqq x \leqq 1$ the transformation is

$$x = \frac{\alpha + 1}{2} - \frac{(\alpha - 1)}{2} y$$

and the integral is

$$T(\alpha, \alpha_{i+1}) = \int_{-1}^{+1} \frac{(P_0 + P_1 y + P_2 y^2 + \cdots)}{\sqrt{1 + y}} dy,$$

where the P_i are constants.

The total period T is

$$T = T(\alpha_i, \alpha) + T(\alpha, \alpha_{i+1}).$$

If the absolute values of two of the e_j are greater than unity then the interval $-1 \leqq x \leqq +1$ can be broken into three intervals and the above process applied. The method is extensible to l such factors.

3·34. Solution of the Differential Equation. Equation (96), by means of a first integration, the change of variable from z to x, and the reductions of §3·33 becomes

$$\frac{dx}{dt} = A[(1 - x^2)(1 + \sigma_1 x)(1 + \sigma_2 x) \cdots$$
$$(1 + \sigma_p x)(1 + d_1 x + e_1 x^2) \cdots (1 + d_l x + e_l x^2) g(x)]^{1/2}. \quad [98a]$$

In Eq. (98a) make the substitutions

$$\sigma_i = \sigma_i' k, \quad d_i = d_i' k, \quad e_i = e_i' k^2, \quad g_i = g_i' k^i,$$

where the g_i are the coefficients in the expansion of $g(x)$. The periods of Eq. (98a) can be obtained by the method of §3·33 and are of the form

$$T = \frac{2\pi}{A} Q(k) = \frac{2\pi}{A}(1 + \bar{Q}_2k^2 + \bar{Q}_4k^4 + \cdots). \qquad [99]$$

To set in evidence the period in question of the solution of (96) and (98a) change the independent variable from t to τ by the substitution $t = Q(k)\tau/A$. Equation (98a) then becomes

$$\frac{dx}{d\tau} = Q(k)[1 - x^2)(1 + \sigma_1'kx) \cdots (1 + \sigma_p'kx)(1 + d_1'kx + e_1'k^2x^2) \cdots$$
$$(1 + d_l'kx + e_l'k^2x^2)(1 + g_1'kx + \cdots + g_s'k^sx^s + \cdots)]^{1/2}. \qquad [100]$$

This equation is of the form of Eq. (2) of this chapter. Consequently there exists a solution of the form

$$x = x_0(\tau) + kx_1(\tau) + k^2x_2(\tau) \cdots$$

which is convergent for k sufficiently small or what is the same thing for $k = 1$ and σ_i', d_i', e_i', g_i' sufficiently small. This solution is periodic of period 2π in τ and of period $T = \frac{2\pi}{A} Q(k)$ in t. The method is employed in non-linear circuit analysis in §3·35.

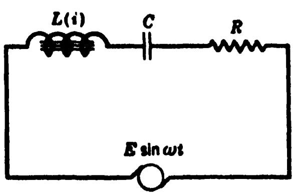

FIG. 3·11. Non-linear Series Circuit.

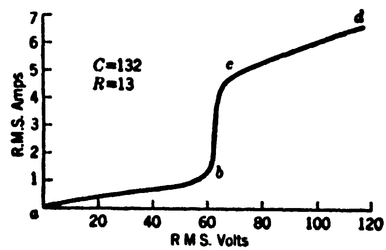

FIG. 3·12. Volt-ampere Characteristic of Non-linear Series Circuit.

3·35. Resonance in Series Non-linear Circuits. The resonance theory of series non-linear control circuits illustrates the principles of Sec. 6. The theory here developed is applicable to series circuits possessing variable inductance, capacitance, and resistance.

Many experimental facts regarding these circuits appear in the literature. The three most pertinent are the following.

(a) *Volt-ampere characteristic.* The volt-ampere characteristic of the circuit in Fig. 3·11 is shown in Fig. 3·12. The values of current and

voltage displayed are root-mean-square values. In the regions *ab* and *cd* the current response is approximately linear with the voltage whereas in the region *bc* the current is critical with respect to the voltage, i.e., a slight increase in voltage produces a large increase in current. The value of the applied voltage for which this increase in current is greatest is called resonant voltage.

(*b*) *Resistance limited.* The maximum value of the current in the region *bc* is resistance limited, i.e., the peak of the current is given by $i = E/R$.

(*c*) *Phase agreement.* At resonant voltage, the voltage and current are nearly in phase.

The *B-H* function is of course many-valued, but this function tends to become single-valued at large magnetizing forces for nicalloy, permalloy, and low-loss steels. Moreover, the numerical integration of the differential equations by means of the integraph shows that the graph obtained for the current in circuits of variable inductance is not changed by employing a single-valued *B-H* function. Accordingly the equation of the magnetization curve is taken to be

$$H = ki = x - a_3x^3 + a_5x^5, \qquad [101]$$

where H = magnetizing force in gilberts per square centimeter,
i = current in amperes,
$x = B/B_0$ where B is the flux density in gausses per square centimeter

and B_0 is the slope of the magnetization curve at the origin. The quantities k, a_3, and a_5 are positive constants. More terms may be added to Eq. (101) if necessary.

The differential equation for the current in the circuit shown in Fig. 3·11 is

$$L(i)\frac{di}{d\tau} + Ri + \frac{1}{C}\int i\,d\tau = -E\cos\omega(\tau - \tau_0), \qquad [102]$$

which, by means of Eq. (101) and the substitution $\theta = \omega\tau$, becomes

$$M\frac{dx}{d\theta} + R(x - a_3x^3 + a_5x^5) + x_c\int(x - a_3x^3 + a_5x^5)d\theta = -Ek\cos(\theta - \theta_0), \qquad [103]$$

where $M = k\omega NAB\,10^{-8}$, C = capacitance in micro-farads,
$x_c = 1/\omega C$, R = resistance in ohms,
$\omega = 377$, A = area of cross-section of coil in square centimeters,
τ = time in seconds, N = number of turns.

Differentiating Eq. (103) we obtain

$$M\frac{d^2x}{d\theta^2} + R(1 - 3a_3x^2 + 5a_5x^4)\frac{dx}{d\theta} + x_c(x - a_3x^3 + a_5x^5) = Ek\sin(\theta - \theta_0). \quad [104]$$

To investigate the resonance between the applied voltage and the circuit it is necessary to determine the natural period of the circuit, i.e., it is necessary to integrate Eq. (104) for $E = 0$. This integration is accomplished in two steps. First Eq. (104) is integrated for $E = R = 0$ and then the solution is modified to take care of the resistance. Accordingly, the first equation to be integrated is

$$M\frac{d^2x}{d\theta^2} + x_c(x - a_3x^3 + a_5x^5) = 0. \quad [105]$$

Performing a first integration of (105) we have

$$\left(\frac{dx}{d\theta}\right)^2 = \frac{a_5x_c}{3M}\left(\frac{3}{a_5}C_o^2 - \frac{3}{a_5}x^2 + \frac{3a_3}{2a_5}x^4 - x^6\right), \quad [106]$$

where

$$\left(\frac{dx}{d\theta}\right)^2 = \frac{x_c}{M}C_o^2 \quad \text{for } x = 0.$$

The integral of Eq. (106) is hyperelliptic and we obtain first the period of the solution. The period of the solution is dependent, as in the elliptic case, on the amplitude of the flux.

The right side of Eq. (106) vanishes for only one real value of x, i.e., at the maximum value of c of the flux density. Accordingly, let (106) be written

$$\left(\frac{dx}{d\theta}\right)^2 = \frac{a_5x_c}{3M}(c^2 - x^2)[x^4 + 2(b^2 - a^2)x^2 + (a^2 + b^2)^2] = 0, \quad [107]$$

where $\quad (a^2 + b^2)^2 = \dfrac{3C_o^2}{a_5c^2}, \; (a^2 + b^2)^2 - 2(b^2 - a^2)c^2 = \dfrac{3}{a_5}$

$$\begin{aligned} c^2 - 2(b^2 - a^2) &= \frac{3a_3}{2a_5}, \; C_o^2 = c^2\left(1 - \frac{a_3}{2}c^2 + \frac{a_5}{3}c^4\right), \\ a^2 &= \frac{1}{2}\left(c^4 - \frac{3a_3}{2a_5}c^2 + \frac{3}{a_5}\right)^{1/2} - \frac{1}{4}\left(c^2 - \frac{3a_3}{2a_5}\right), \\ b^2 &= \frac{1}{2}\left(c^4 - \frac{3a_3}{2a_5}c^2 + \frac{3}{a_5}\right)^{1/2} + \frac{1}{4}\left(c^2 - \frac{3a_3}{2a_5}\right). \end{aligned} \quad [108]$$

If $x = cy$, $a/c = \beta$, and $b/c = \gamma$, Eq. (107) becomes

$$\left(\frac{dy}{d\theta}\right)^2 = A_o^2\, c^4(1 - y^2)[(\beta^2 + \gamma^2)^2 + 2(\gamma^2 - \beta^2)y^2 + y^4], \quad [109]$$

where $A_0^2 = a_5 x_c/3M$.

The solution of Eq. (109) possesses but one real period T_c which is

$$T_c = \frac{2}{c^2 A_o} \int_{-1}^{+1} \frac{dy}{\{(1 - y^2)[(\beta^2 + \gamma^2)^2 + 2(\gamma^2 - \beta^2)y^2 + y^4]\}^{1/2}}$$

$$= \frac{2}{c^2 A_o} (T_1 + T_2 + T_3),$$

where

$$T_1 = \int_{-1}^{-\beta} \frac{\{[(y - \beta)^2 + \gamma^2][(y + \beta)^2 + \gamma^2]\}^{-1/2}}{\sqrt{(1 - y^2)}}\, dy,$$

$$T_2 = \int_{-\beta}^{\beta} \frac{\{[(y - \beta)^2 + \gamma^2][(y + \beta)^2 + \gamma^2]\}^{-1/2}}{\sqrt{(1 - y^2)}}\, dy,$$

$$T_3 = \int_{\beta}^{1} \frac{\{[(y - \beta)^2 + \gamma^2][(y + \beta)^2 + \gamma^2]\}^{-1/2}}{\sqrt{(1 - y^2)}}\, dy.$$

Evidently, $T_1 = T_3$. By the substitution

$$y = \frac{1 - \beta}{2}\eta - \frac{1 + \beta}{2},$$

T_1 is reduced to

$$T_1 = \int_{-1}^{+1} \frac{4(1 - \beta)^{1/2}\{[1 - 2x_1h_1 + h_1^2][1 - 2x_2h_2 + h_2^2](1 - a_0\eta)\}^{-1/2}}{\{(3 + \beta)[(1 + 3\beta)^2 + 4\gamma^2][(1 - \beta)^2 + 4\gamma^2](1 + \eta)\}^{1/2}}\, d\eta, \quad [110]$$

where

$$h_1 = \frac{(1 - \beta)\eta}{[(1 + 3\beta)^2 + 4\gamma^2]^{1/2}} = a_1\eta, \quad x_1 = \frac{1 + 3\beta}{[(1 + 3\beta)^2 + 4\gamma^2]^{1/2}},$$

$$h_2 = \frac{(1 - \beta)\eta}{[(1 - \beta)^2 + 4\gamma^2]^{1/2}} = a_2\eta, \quad x_2 = \frac{1 - \beta}{[(1 - \beta)^2 + 4\gamma^2]^{1/2}},$$

$$a_0 = \frac{1 - \beta}{3 + \beta}.$$

Since $0 < h_i < 1$, $0 < x_i < 1$

$$(1 - 2x_ih_i + h_i^2)^{-1/2} = 1 + h_iP_1(x_i) + h_i^2P_2(x_i) + \cdots + h_i^nP_n(x_i) \quad (i = 1, 2),$$

where $P_1, P_2, \cdots, P_n$ are Legendre polynomials. The expression $(1 - a_0\eta)^{-1/2}$ is expansible in a rapidly convergent series. Substitution of these expressions in Eq. (110) and the carrying out of the integrations yield $T_1 + T_3 = 2\pi Q_0 S_0$, where

$$Q_0 = \frac{8\sqrt{2a_0}}{\pi\{[(1+3\beta)^2 + 4\gamma^2][(1-\beta)^2 + 4\gamma^2]\}^{1/4}}$$

$$S_0 = 1 - \tfrac{1}{3}(A_0' + \tfrac{1}{2}a_0) + \tfrac{7}{15}(\tfrac{3}{8}a_0^2 + \tfrac{1}{2}a_0A_0' + A_2) + \cdots,$$

$$A_0' = a_1x_1 + a_2x_2,$$

$$A_2 = a_1^2P_1(x_1) + a_1a_2P_1(x_1)P_1(x_2) + a_2^2P_2(x_2),$$

$$A = c^2A_0.$$

By means of the substitution $y = \beta\eta$ and the method employed in evaluating T_1 and T_3, the value of T_2 is $T_2 = \pi Q_1 S_1$ where

$$Q_1 = \frac{2\beta}{\pi(\beta^2 + \gamma^2)},$$

$$S_1 = 1 + \tfrac{1}{3}(2a_3^4 - a_3^2 + \tfrac{1}{2}\beta^2) + \tfrac{1}{10}\beta^2(2a_3^4 - a_3^2) + \cdots,$$

$$a_3 = \frac{\beta}{(\beta^2 + \gamma^2)^{1/2}}.$$

The series S_0 and S_1 are so rapidly convergent that two terms are sufficient in all computations. Finally,

$$T_c = \frac{2\pi}{A}(Q_0S_0 + Q_1S_1) = \frac{2\pi}{A}Q(\beta^2, \gamma^2) = \frac{2\pi}{A}Q.$$

The equation is now integrated for zero resistance. Differentiating Eq. (109) with respect to θ, canceling out $\frac{dy}{d\theta}$, changing the independent variable from θ to t by means of the substitution $\theta = \frac{Qt}{A} = \frac{(1+\delta)}{A}t$, and writing $\frac{d^2y}{dt^2} = y''$, we have

$$y'' + (1+\delta)^2\{[(\beta^2+\gamma^2)^2 - 2(\gamma^2 - \beta^2)]y + 2[2(\gamma^2 - \beta^2) - 1]y^3 + 3y^5\} = 0. \quad [111]$$

Equation (111) by the aid of the identities (108) can be written

$$y'' + (1+\delta)^2\{[1 + a_{12}\mu^2 + a_{14}\mu^4]y + [a_{32}\mu^2 + a_{34}\mu^4]y^3 + a_{54}\mu^4y^5\} = 0 \quad [112]$$

where

$$a_{12} = -2k_0^2, \qquad a_{14} = \left(1 - \frac{a_5}{3c_0^2}c^6\right)k^4, \quad a_{52} = 0,$$

$$a_{32} = 4k_0^2, \qquad a_{34} = -\tfrac{2}{3}a_5\frac{c^6}{c_0^2}k^4, \qquad a_{54} = \frac{a_5c^6k^4}{c_0^2}, \qquad [113]$$

$$k_0^2 = \gamma^2 - \beta^2, \quad k^2 = \beta^2 + \gamma^2,$$

and μ is a parameter to guide the integration. The parameter μ is subsequently set equal to unity. If the core of the reactor is any kind of magnetic material with a moderately sharp B-H curve the ranges of the constants employed either directly or indirectly turn out to be

$$2 < c < 5, \quad 2000 < B_0 < 5000, \quad 0.4 < \beta < 0.7,$$

$$0.4 < \gamma < 0.8, \quad 0.3 < a_3 < 3, \quad 0.01 < a_5 < 0.5,$$

$$0 < k_0 < 1, \quad 0 < k < 1, \quad -1 < \delta < +1.$$

Now the solution of Eq. (112) is periodic of period T in θ and of period 2π in t. This solution is now obtained. By §3·2 there exists a solution of (112) of the form

$$y = y_0 + y_2\mu^2 + y_4\mu^4 + \cdots, \qquad [114]$$

which converges for μ^2 sufficiently small, or which converges for $\mu^2 = 1$, provided both k_0^2 and k^2 are sufficiently small. Write

$$\delta = \delta_2\mu^2 + \delta_4\mu^4 + \cdots. \qquad [115]$$

Substituting Eqs. (114–115) in Eq. (112) and equating like powers of μ^2 we obtain

$$\begin{aligned} &y_0'' + y_0 = 0, \\ &y_2'' + y_2 + 2\delta_2y_0 + a_{12}y_0 + a_{32}y_0^3 = 0, \\ &y_4'' + y_4 + (\delta_2^2 + 2\delta_4)y_0 + a_{14}y_0 + a_{34}y_0^3 + a_{54}y_0^5 \\ &\qquad + a_{12}y_2 + 3a_{32}y_0^2 + 2\delta_2(a_{12}y_2 + a_{32}y_0^3) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad [116]$$

for the determination of $y_0, y_2, y_4 \cdots$.

To determine the initial conditions in t, substitute $\theta = \dfrac{(1+\delta)t}{A}$ in Eq. (109) and let $y = 0$. Then

$$y'(0) = (1+\delta)(\beta^2 + \gamma^2) = (1 + \delta_2\mu^2 + \delta_4\mu^4 + \cdots)A_{11} \qquad [117]$$

from which $y_0'(0) = A_{11}$. The solution of the first of Eqs. (116) is

$$y_0 = A_{11}\sin t.$$

When this value of y_0 is substituted in the second of Eqs. (116) it follows that the solution for y_2 cannot be periodic unless δ_2 is so chosen that the linear terms in y_0 vanish. From this condition

$$\delta_2 = (1 - \tfrac{3}{2}A_{11}^2)k_0^2.$$

The initial condition for y_2 is $y_2'(0) = \delta_2 A_{11}$. Thus the solution for y_2 is

$$y_2 = A_{21} \sin t + A_{23} \sin 3t,$$

where

$$A_{21} = A_{11}k_0^2(1 - \tfrac{15}{8}A_{11}^2), \quad A_{23} = \tfrac{1}{8}k_0^2A_{11}^3.$$

When y_0 and y_2 are substituted in the third equation of (116) the coefficient of the linear terms in y_0 must vanish in order that y_4 be periodic in t. Thus the δ's and y's may be found sequentially as far as desired.

Owing to the arrangement of the problem the maximum value of y is unity and the maximum value of the instantaneous flux is B_0c. The flux B is given by the relation $B = B_0cy$. Substitution in Eq. (101) gives the instantaneous value of i.

The solution can now be extended so as to include resistance. The solution of Eq. (104) for $E = 0$ is, of course, not periodic. The solution decays; its period increasing and its amplitude decreasing as c diminishes until the circuit has become linear. The solution after the circuit has become linear is, of course, a damped sinusoid of fixed period. But we are interested eventually in those cases where the applied voltage $E \sin(\theta - \theta_0)$ maintains in the steady-state the maximum value of B at B_0c. From physical considerations it is obvious that the time of oscillation is increased by resistance. From Eq. (104) it is evident that the effect of resistance on the flux is a non-linear one.

It is now desired to obtain the natural period of the circuit with resistance when the circuit is operated at maximum flux B_0c. By means of Eqs. (108) and the relation $\theta = \dfrac{Q}{A}(1 + \delta)t$, Eq. (104) can be transformed to

$$y'' + r(1 + 3b_3y^2 + 5b_5y^4)y' + (1 + \delta)^2(b_1'y + b_3'y^3 + b_5'y^5) = 0, \quad [118]$$

where

$$r = \frac{R}{x_c}AQ(1 + a_{12}\mu^2 + a_{14}\mu^4) = \frac{R}{x}AQb_1',$$

$$b_1' = k^4 - 2k_0^2, \quad b_3' = 2(2k_0^2 - 1), \quad b_3' = 3,$$

$$b_3 = \frac{b_3'}{b_1'}, \quad b_5 = \frac{b_5'}{b_1'}.$$

Equation (118), written in the normal form, is

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -r(1 + 3b_3y^2 + 5b_5y^4)y_2 - (1+\delta)^2(b_1'y_1 + b_3'y_1^3 + b_5'y_1^5). \end{aligned} \tag{119}$$

The solution of system (119) for $r = 0$ is

$$y = (A_{11} + A_{21}) \sin t + A_{23} \sin 3t + \cdots.$$

On passing to numerical results y is observed to be practically sinusoidal. Thus let the solution of Eqs. (119) for $r = 0$ be

$$\begin{aligned} y_1 &= u \sin (t + v), \\ y_2 &= u \cos (t + v). \end{aligned} \tag{120}$$

In completing the solution of Eq. (119) we shall employ the method of differential variation of parameters. (See Sec. 2, this chapter.) Evidently,

$$\begin{aligned} \frac{\partial y_1}{\partial u}\frac{du}{dt} + \frac{\partial y_1}{\partial v}\frac{dv}{dt} + \frac{\partial y_1}{\partial t} &= y_2, \\ \frac{\partial y_2}{\partial u}\frac{du}{dt} + \frac{\partial y_2}{\partial v}\frac{dv}{dt} + \frac{\partial y_2}{\partial t} &= y_2'. \end{aligned} \tag{121}$$

Equations (121) yield

$$\begin{aligned} u' &= -r(1 + 3b_3y_1^2 + 5b_5y_1^4)u \cos^2(t + v), \\ v' &= +r(1 + 3b_3y_1^2 + 5b_5y_1^4) \sin (t + v) \cos (t + v). \end{aligned} \tag{122}$$

By substituting Eqs. (120) in (122) and carrying out the expansions we obtain

$$\begin{aligned} u' &= -r[e_0 + e_2 \cos (t + v) + e_4 \cos 4(t + v) + e_6 \cos 6(t + v)], \\ v' &= +r[f_2 \sin 2(t + v) + f_4 \sin 4(t + v) + f_6 \sin 6(t + v)], \end{aligned} \tag{123}$$

where

$$\begin{aligned} e_0 &= \tfrac{1}{2}u + \tfrac{3}{8}b_3u^3 + \tfrac{5}{16}b_5u^5, & f_2 &= \tfrac{1}{2} + \tfrac{3}{4}b_3u^3 + \tfrac{25}{32}b_5u^4, \\ e_2 &= \tfrac{1}{2}u - \tfrac{5}{32}b_5u^5, & f_4 &= -\tfrac{3}{8}b_3u^2 - \tfrac{5}{8}b_5u^4, \\ e_4 &= -\tfrac{3}{8}b_3u^3 - \tfrac{5}{16}b_5u^5, & f_6 &= \tfrac{5}{32}b_5u^4, \end{aligned}$$
$$e_6 = \tfrac{5}{32}b_5u^5.$$

By §3·8 there exists a solution of Eqs. (123) as a power series in r. It is clear that $u = (A_{11} + A_{21}) = e$ and $v = 0$ for $r = 0$. Accordingly, write

$$\begin{aligned} u &= e_0 + u_1r + u_2r^2 + \cdots \\ v &= v_1r + v_2r^2 + \cdots. \end{aligned} \tag{124}$$

Now r, for control circuits, is very small. Consequently, v is small. Expanding $\cos n(t + v)(n = 2, 4, 6)$ in Eqs. (123) as a power series in v, substituting Eqs. (124) in (123), and equating, in the resulting equations, the coefficients of like powers of r we obtain

$$\begin{aligned} u_0' &= 0, \\ v_1' &= f_2 \sin 2t + f_4 \sin 4t + f_6 \sin 6t, \\ u_1' &= -\,(e_0 + e_2 \cos 2t + e_4 \cos 4t + e_6 \cos 6t), \\ v_2' &= (g_2 \sin 2t + g_4 \sin 4t + g_6 \sin 6t)u \\ &\qquad + 2v_1(f_2 \cos 2t + 2f_4 \cos 4t + 4f_6 \cos 6t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{125}$$

where

$$\begin{aligned} g_2 &= (\tfrac{3}{2}b_3 + \tfrac{25}{8}b_5 e^2)e, \\ g_4 &= -\,(\tfrac{3}{4}b_3 + \tfrac{5}{2}b_5 e^2)e, \\ g_6 &= \tfrac{5}{8}b_5 e^3. \end{aligned}$$

Equations (125) are solved sequentially subject to the initial conditions $u_0 = e$, $u_i = v_i = 0$ for $t = 0$, $i = 1, 2, \cdots$. The solutions are

$$\begin{aligned} u_0 &= e, \\ v_1 &= \tfrac{1}{2}[f_2(1 - \cos 2t) + \tfrac{1}{2}f_4(1 - \cos 4t) + \tfrac{1}{3}f_6(1 - \cos 6t)], \\ u_1 &= -\,(e_0 t + \tfrac{1}{2}e_2 \sin 2t + \tfrac{1}{4}e_4 \sin 4t + \tfrac{1}{6}e_6 \sin 6t), \\ v_2 &= \tfrac{1}{2}e_0 t(g_2 \cos 2t + \tfrac{1}{2}g_4 \cos 4t + \tfrac{1}{3}g_6 \cos 6t) - \\ &\qquad \tfrac{1}{4}t(g_2e_2 + \tfrac{1}{2}g_4e_4 + \tfrac{1}{3}g_6e_6) + \tfrac{1}{2}(f_2^2 + f_4^2 + f_6^2)t, \end{aligned} \tag{126}$$

where the f's, e's, and g's of Eqs. (126) are the same symbols previously defined, except u is replaced by e.

Substituting Eqs. (126) in (124) and (124) in turn in (120) the solution of Eqs. (118) as far as the terms in r^2 is obtained. The immediate objective is to find the period of a half-cycle when the applied voltage maintains the circuit in operation with maximum flux B_0c. It is then necessary to solve $y_1(t) = 0$ for t. Evidently y_1 vanishes for $t + v_1(t)r + v_2(t)r^2 + \cdots = \pi$. A solution of this equation by Newton's method (an approximate root is π) is

$$t = \pi(1 + \frac{r^2}{8}\delta_0), \tag{127}$$

where

$$\delta_0 = 1 + 2.24b_3e^2 + (2.08b_5 + 1.4b_3^2)e^4 + 0.36b_3b_3e^6 + 1.4b_5^2e^8. \tag{128}$$

Finally, by means of the transformation $\theta = \frac{Q}{A} t$ and equations $T_c = \frac{2\pi}{A} Q$ and (127) the period, taking into account resistance, is

$$T_{cr} = \frac{2\pi}{A} Q \left(1 + \frac{r^2}{8} \delta_0\right). \qquad [129]$$

The period in θ of the applied voltage, Eq. (104), is 2π. The condition for the circuit to be in resonance with the applied voltage is $T = 2\pi$. Substituting this value in Eq. (129), neglecting the term containing r^4, and solving for x_c we obtain

$$x_c = \frac{3M Q^2}{2c^4 a_5} \left(1 + \sqrt{1 + \tfrac{1}{9} \delta_0 c^8 a_5^2 b_1'^2 M^{-2} R^2}\right). \qquad [130]$$

The final results are: (*a*) proof of the physical principle that the sudden increase in current in the region *bc* (Fig. 3·12) is due to the circuit being in resonance with the applied voltage, (*b*) formula (130) giving the amount of capacitative reactance required to produce resonance at a prescribed voltage. The theory checks accurately experimental results.[23]

EXERCISES XX

1. Prove that the equation

$$\frac{d^2x}{dt^2} = -(1 + k^2 + k_0^2)x + 2(k^2 + k_0^2 + k^2k_0^2)x^3 - 3k^2k_0^2x^5,$$

has two fundamental real periods and one imaginary period. Taking the initial conditions to be $x(0) = 0$, $x'(0) = 1$, and $0 < k^2 < k_0^2 < 1$, obtain these periods.

2. If a hemisphere rocks so that its motion remains in a plane, the differential equation of motion is

$$(r^2 + n^2 - 2rn \cos \theta)\ddot{\theta} + gn \sin \theta = 0,$$

where g is the acceleration of gravity, and the remaining quantities are shown on Fig. 3·13. Integrate the differential equation subject to the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$.

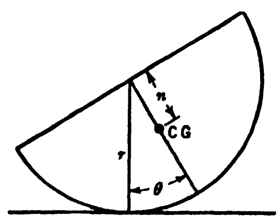

FIG. 3·13. Rocking Hemisphere.

3. Solve the non-linear spring problem of §3·31, taking as the expression for the force F

$$F = b_1x + b_3x^3 + b_5x^5.$$

Let b_3 and b_5 be large compared with b_1.

[23] E. G. Keller, "Resonance Theory of Series Non-linear Control Circuits," *J. Franklin Institute*, **225**, 561–577 (1938).

4. Let the non-linear spring of §3·31 be subject to a periodic force $a \cos \omega t$ such that the differential equation of motion of m is

$$m \frac{d^2x}{dt^2} + (b_1 x + b_3 x^3) = a \cos \omega t.$$

Deduce the condition for resonance between the applied force and the natural period.

PROBLEMS XXI

1. In Ex. 4 above let ω have a value such that neither resonance nor beats occur in the motion of m. Under these conditions obtain a solution of the differential equation of Ex. 4 which will give the amplitude of the motion. There is an electrical analogue.[24]

2. Set up the differential equations for the primary and secondary currents of a transformer where the saturation curve is $ki = x - a_3x^3 + a_5x^5$ and $x = B/B_0$. The primary impressed voltage is $E \sin t$ and there is a condenser in both the primary and secondary circuits.

3. The differential equation of a simple series circuit with sinusoidal impressed voltage, constant inductance, and thyrite resistance is

$$M \frac{di}{d\theta} + R(i)\, i = E \sin \theta,$$

where $R(i) = K\, i^{-71/101}$ and K is a constant. Solve this equation.

4. Read[25] the paper listed below and then by means of a non-linear inductive circuit with thyrite resistance, design a lightning arrester such that when the current reaches its peak in the inductance the resistance of the thyrite element is near zero.

3·36. Advanced Schwarzian Transformations. The theory of elliptic functions is employed in the study of two-dimensional field problems[26] by means of the Schwarzian transformation. More advanced Schwarzian transformations become hyperelliptic. Even these can be carried out if the hyperelliptic functions are expressed in the form of those of §3·32.

(7)

Method of Collocation

The method of collocation is primarily one of solving systems of linear differential equations. However, it is extensible by means of

[24] E. G. Keller, "Beat Theory of Non-Linear Circuits," *J. Franklin Institute*, **228** (September, 1939).

[25] K. B. McEachron, "Thyrite: A New Material for Lightning Arresters," *General Electric Review*, February, 1930, p. 92.

[26] I. A. Terry and E. G. Keller, "Field-Pole Leakage Flux in Salient-Pole Dynamo-Electric Machines," *Journal of the Institution of Electrical Engineers*, **83**, 845–854 (1938).

implicit function theory (§ 3·12) to non-linear systems. Essentially, the method consists in setting up a sequence of functions which satisfy precisely the boundary conditions of a system of differential equations and which satisfy the differential equations to a prescribed degree of approximation. The prescribed degree of approximation is specified in the next article. The success of the application of the method is determined largely by the skillful choice of the functions which form the sequence. The guide in the choice of the functions is a thorough knowledge of the physical problem.

3·37. Theory of Method of Collocation. In §3·1 it was indicated that any system of differential equations can be reduced to the normal form of Eqs. (1). It is evident also that any system of differential equations may be reduced, by repeated differentiations and elimination of variables, to a single differential equation of the same order as the order of the system. Accordingly, in the method of collocation, we shall take as the normal form of a system of linear differential equations the single nth order differential equation

$$f_0(x)\frac{d^n y}{dx^n} + f_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + f_n(x)y - \varphi(x) = 0, \qquad [131]$$

or

$$f(p)y - \varphi(x) = 0,$$

where

$$f(p)y \equiv [f_0(x)p^n + f_1(x)p^{n-1} + \cdots + f_n(x)]y.$$

Let the interval of the solution, i.e., the range of the independent variable be $a \leqq x \leqq b$. The boundary conditions are expressed in a form different from that used in the classical solution of §§3·2, 3·9, and 3·14. In the present method the boundary conditions consist of the n equations

$$f_0(x_0)p^n y\big|_{x=x_0} + \cdots + f_n(x_0)y(x_0) = B_i \quad (i = 1, 2, \cdots, n), \qquad [132]$$

where $f_j(x_0)$ and B_i are constants and not all B_i are zero. The n values of x_0 satisfy the relation $a \leqq x_0 \leqq b$. The n values of x_0 are not necessarily identical for the n equations (132).

The first part of the construction of the solution consists in setting up $s + 1$ linearly independent functions $Y_0, Y_1, \cdots, Y_s$ of x such that $y = Y_0(x)$ satisfies Eqs. (132) where the B_i have definite values not all of which are zero and each of the functions $Y_1, Y_2, \cdots, Y_s$ satisfies (132) with every B_i replaced by zero. Evidently then the function

$$Y = Y_0 + \sum_{j=1}^{s} c_j Y_j, \qquad [133]$$

where the c_j are arbitrary constants, will satisfy the boundary conditions (132). In a physical problem the $Y_j (j = 0, 1, 2, \cdots, s)$ are determined by principles of physics and engineering. The functions must be such that a linear combination of them will permit the behavior of the dependent variable anticipated. Experience with the system, or oscillographic or differential analyzer solutions of the equations may serve as a guide in the choice of the Y_j.

The second part of the construction of the solution consists in determining the c_j such that Eq. (133) will satisfy (131) at least approximately. Denote by $\epsilon(x)$ the result of substituting Y of (133) for y in (131). The result of the substitution is

$$\begin{aligned} \epsilon(x) &= f(p)Y(x) - \varphi(x) \\ &= \sum_{j=1}^{s} c_j Z_j + Z_0 - \varphi(x), \end{aligned} \tag{134}$$

where

$$Z_j = f(p)Y_j \quad (j = 1, 2, \cdots, s).$$

In the method of collocation the condition is imposed that the c_j shall be such that $\epsilon(x)$ shall be zero at s different values of x. [That is, the exact solution of (132) and the approximate solution (133) shall have identical *locations* at s points.] From this condition Eq. (134) becomes the s equations

$$\sum_{j=1}^{s} Z_j(x_k)c_j = \varphi(x_k) - Z_0(x_k), \quad (k = 1, 2, \cdots, s). \tag{135}$$

Since Eqs. (135) are linear in c_j the s equations are readily solvable for c_j. When the values of c_j so determined are substituted in Eq. (133) the value of Y there defined is the approximate solution desired. It is frequently convenient to solve for c_j by matrices.

EXAMPLE. The method of collocation is illustrated by the solution of $\frac{dy}{dx} - y = 0$ subject to the initial conditions $y = 1$ for $x = 0$. Let the interval of the solution be $0 \leqq x \leqq 1$. If the choices are made that

$$Y_0 = 1, \quad Y_j = x^j \quad (j = 1, 2, 3, 4)$$

then

$$Y = Y_0 + \sum_{j=1}^{4} c_j Y_j = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4,$$

and

$$Z_0 = -1, \quad Z_j = \left(\frac{d}{dx} - 1\right) x^j = j\,x^{j-1} - x^j, \quad \varphi(x) = 0.$$

Since there are four unknown c's it is necessary to use four values of x_k. Let these be $x_1 = 0$, $x_2 = \frac{1}{3}$, $x_3 = \frac{2}{3}$, and $x_4 = 1$. Equations (135) now become, in matrix notation,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{5}{9} & \frac{8}{27} & \frac{11}{81} \\ \frac{1}{3} & \frac{8}{9} & \frac{28}{27} & \frac{80}{81} \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The solution of these equations is $c_1 = 1$, $c_2 = 0.5078$, $c_3 = 0.1406$, $c_4 = 0.0703$. The approximate solution of the differential equation is

$$Y = 1 + x + 0.5078x^2 + 0.1406x^3 + 0.0703x^4.$$

EXERCISES XXII

1. Solve by the method of collocation $(p^2 + 1)y = 0$ subject to the initial conditions $y(0) = 0$, $y'(0) = 1$. Let the interval, in which the solution is desired, be $-\pi \leqq x \leqq \pi$. The functions Y_0 and Y_j are suggested by

$$Y = Y_0 + c_1Y_1 + c_2Y_2 = x(\pi^2 - x^2)(1/\pi^2 + c_1x^2 + c_2x^4).$$

2. Solve exercise 1 by employing $Y_0 = \sin x$, $Y_1 = \cos x$ so that $Y = \sin x + c_1 \cos x$.

3. Solve, by the method of collocation, Legendre's equation

$$\frac{d^2y}{dx^2} - \frac{2x}{1 - x^2}\frac{dy}{dx} + \frac{m(m + 1)y}{1 - x^2} = 0$$

subject to the initial conditions $y(0) = -\frac{1}{2}$, $y'(0) = 0$. Let $m = 2$ and let the interval for the solution be $0 \leqq x \leqq 0.5$. Compare the accuracy and the total labor done in obtaining the solution with that done for the matrix solution of § 3·23.

3·38. Method of Collocation for a Non-linear Problem.

The method of collocation is applicable to non-linear problems. Let it be required to solve $dy/dx = y^2/2$ subject to the initial condition $y(0) = 1$. [The exact solution is $y = (1 - x/2)^{-1}$.] Let

$$Y = Y_0 + \sum_{j=1}^{s} c_jY_j = 1 + c_1x + c_2x^2 + \cdots + c_sx^s.$$

Substituting Y in the differential equation we have

$$c_1 + 2c_2x_k + \cdots + s\,c_sx_k^{s-1} = \tfrac{1}{2}(1 + c_1x_k + \cdots + c_sx_k^s)^2,$$

where $k = 1,2, \cdots, s$. If $s = 3$ and $x_1 = 0$, $x_2 = 0.5$, $x_3 = 1$, the last equations become

$$c_1 = 0.5,$$

$$\frac{1}{2} + c_2 + \frac{3c_3}{4} = \frac{1}{2}\left(\frac{5}{4} + \frac{c_2}{4} + \frac{c_3}{8}\right)^2,$$

$$\frac{1}{2} + 2c_2 + 3c_3 = \frac{1}{2}\left(\frac{3}{2} + c_2 + c_3\right)^2.$$

One solution of the above equations is $c_1 = 0.5$, $c_2 = -0.0207$, $c_3 = 0.5004$. A closely approximate solution of the differential equation is

$$y = 1 + 0.5x - 0.0207x^2 + 0.5004x^3.$$

EXERCISES XXIII

1. Solve, by the method of collocation, the differential equation $\frac{d^3y}{dx^3} = y^2$, subject to the initial conditions $y(0) = y'(0) = y''(0) = 1$.

(8)

Galerkin's Method

Galerkin's method is of special value in the solution of problems in dynamics and elasticity. The method differs from the method of collocation only in the conditions imposed on $\epsilon(x)$. [See Eq. (134) for definition of $\epsilon(x)$.]

3·39. The Galerkin Equations. Galerkin's theory is identical to the theory explained in §3·37 as far as Eq. (134). The interval for which the solution of the differential equation is sought is $a \leqq x \leqq b$. Either or both a and b can be infinite. The condition imposed on $\epsilon(x)$ is that

$$J = \int_a^b [\epsilon(x)]^2\, dx$$

shall be a minimum. A necessary condition for J to be a minimum is

$$\frac{\partial J}{\partial c_k} = 0, \quad (k = 1, 2, \cdots, s).$$

We shall show that the s equations above reduce to s equations of the form

$$\int_a^b \epsilon(x)\, Y_k(x) dx = 0, \quad (k = 1, 2, \cdots, s). \qquad [136]$$

Evidently,

$$\frac{\partial J}{\partial c_k} = 2\int_a^b \epsilon \frac{\partial \epsilon}{\partial c_k}\, dx,$$

where

$$\frac{\partial \epsilon}{\partial c_k} = f_0(x)\frac{d^n Y_k}{dx^n} + f_1(x)\frac{d^{n-1} Y_k}{dx^{n-1}} + \cdots + f_n(x) Y_k,$$

or

$$\frac{\partial J}{\partial c_k} = 2\int_a^b \epsilon(x) Z_k(x)\, dx,$$

where

$$Z_k(x) = f_0(x)\frac{d^n Y_k}{dx^n} + \cdots + f_n(x) Y_k,$$

$$Z_0(x) = f_0(x)\frac{d^n Y_0}{dx^n} + \cdots + f_n(x) Y_0$$

and

$$\epsilon(x) = Z_0(x) - \varphi(x) + \sum_{r=1}^{s} c_r Z_r(x).$$

Since $Z_k(x)$ are expansible in the form

$$Z_k(x) = \sum_{j=1}^{s} g_{kj} Y_j + \eta_k,$$

where the g_{kj} are constants and η_k are remainders in the expansions,

$$\frac{\partial J}{\partial c_k} = 2\int_a^b \epsilon(x)[g_{k1}Y_1 + \cdots + g_{ks}Y_s + \eta_k]\, dx = 0, \quad (k = 1, 2, \cdots, s) \quad [137]$$

Since the $\int_a^b \epsilon\eta_k\, dx$ are negligible for properly chosen Y_k for the physical problem in question and since the Y_k are linearly independent the last equations yield

$$\int_a^b \epsilon(x) Y_1\, dx = 0, \cdots, \quad \int_a^b \epsilon(x) Y_s\, dx = 0. \quad [138]$$

Equations (138) are the required Galerkin equations.

EXAMPLE. Solve, by Galerkin's method, the differential equation $dy/dx - y = 0$ subject to the initial condition $y(0) = 1$. Let the interval for the solution be $0 \leqq x \leqq 1$.

If $Y_0 = 1$, $Y_1 = x$, $Y_2 = x^2$, $Y_3 = x^3$ then

$$Y = 1 + c_1x + c_2x^2 + c_3x^3$$

and

$$\epsilon(x) = \frac{dY}{dx} - Y = (c_1 - 1) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 - c_3x^3.$$

The Galerkin equations, for the present problem, are

$$\int_0^1 \epsilon x \, dx = \int_0^1 \epsilon x^2 \, dx = \int_0^1 \epsilon x^3 \, dx = 0.$$

The substitution of the value of $\epsilon(x)$ and the carrying out of the indicated integrations yield

$$\begin{aligned} 10c_1 + 25c_2 + 33c_3 &= 30, \\ 5c_1 + 18c_2 + 26c_3 &= 20, \\ 21c_1 + 98c_2 + 150c_3 &= 105. \end{aligned}$$

The values of c_1, c_2, and c_3 are respectively 1.03,0.388,0.301 and the approximate solution is

$$y = Y = 1 + 1.034x + 0.388x^2 + 0.301x^3.$$

3·40. Torsional Oscillations of a Uniform Cantilever by Galerkin's Method. The partial differential equation governing the torsional motion of a uniform cantilever is

$$\frac{\partial}{\partial x}\left(\frac{C\partial\theta}{\partial x}\right) = \frac{I\partial^2\theta}{\partial t^2}, \qquad [139]$$

where θ = angle of twist per unit length,
C = torsional stiffness per unit length,
I = moment of inertia per unit length,
t = time.

Suppose C and I are constant throughout the length of the shaft. If θ is set equal to $X \cdot T$, where X is a function of x alone and T is a function of t alone, and $\theta = X \cdot T$ is substituted in Eq. (139) then

$$\frac{CT}{I}\frac{d^2X}{dx^2} = X\frac{d^2T}{dt^2}$$

or

$$\frac{C}{I}\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2}.$$

The left-hand member of the last equation is a function of x alone and the right-hand member is a function of t alone. This equality holds for

infinitely many values of x and t. Consequently, each member of the equation is equal to a constant and we have

$$\frac{C}{I}\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} = -k^2$$

or

$$\frac{d^2T}{dt^2} + k^2T = 0, \tag{140}$$

$$\frac{d^2X}{dx^2} + \frac{k^2I}{C}X = 0, \tag{141}$$

where $-k^2$ is a constant yet to be determined.

The partial differential equation (139) has thus been reduced to the two ordinary differential equations (140) and (141). The general solution of (140) is

$$T = A \sin kt + B \cos kt,$$

where A and B are arbitrary constants. Likewise the general solution of Eq. (141) can be written down at once. However, we shall solve it by Galerkin's method. By the substitution $x = l\xi$ and $\frac{Ik^2l^2}{C} = m^2$ Eq. (141) and the boundary conditions become respectively

$$\frac{d^2X}{d\xi^2} + m^2X = 0, \quad X(0) = 0, \quad \frac{dX}{d\xi} = 0 \quad \text{for } \xi = 1.$$

First it is necessary to choose functions which satisfy the boundary conditions. A binomial in ξ seems a reasonable approximation to the possible displacement in X. Accordingly, set

$$X_r = (r + 1)\xi^r - r\xi^{r+1}.$$

Evidently, X_r satisfies the boundary conditions for all r. For $r = 1, 2,$

$$X = c_1X_1 + c_2X_2 = c_1(2\xi - \xi^2) + c_2(3\xi^2 - 2\xi^3).$$

The Galerkin equations (Eqs. 138) for the problem under consideration are

$$c_1\int_0^1 X_1\left(\frac{d^2X_1}{d\xi^2} + m^2X_1\right)d\xi + c_2\int_0^1 X_1\left(\frac{d^2X_2}{d\xi^2} + m^2X_2\right)d\xi = 0,$$

$$c_1\int_0^1 X_2\left(\frac{d^2X_1}{d\xi^2} + m^2X_1\right)d\xi + c_2\int_0^1 X_2\left(\frac{d^2X_2}{d\xi^2} + m^2X_2\right)d\xi = 0,$$

which, on substitution of the binomials above and subsequent integration, become

$$c_1(-\tfrac{4}{3} + \tfrac{8}{15}m^2) + c_2(-1 + \tfrac{13}{20}m^2) = 0,$$

$$c_1(-1 + \tfrac{13}{30}m^2) + c_2(-\tfrac{6}{5} + \tfrac{13}{35}m^2) = 0.$$

The necessary and sufficient condition that the above homogeneous system possess a non-trivial solution is that the determinant

$$\Delta = \begin{vmatrix} -\tfrac{4}{3} + \tfrac{8}{15}m^2 & -1 + \tfrac{13}{20}m^2 \\ -1 + \tfrac{13}{30}m^2 & -\tfrac{6}{5} + \tfrac{13}{35}m^2 \end{vmatrix}$$

shall vanish. The roots of $\Delta = 0$ are $m^2 = 2.4680$ and 23.5625. From the homogeneous linear system we have

$$c_2 = \frac{(+1.33 - \tfrac{8}{15}m^2)}{-1 + \tfrac{13}{20}m^2}\, c_1.$$

From this relation $c_2 = 0.0281c_1$ for $m^2 = 2.4680$ and $c_2 = -\,0.77c_1$ for $m^2 = 23.5625$.

Two particular solutions or natural modes of vibration satisfying the partial differential equation are

$$\theta = T\cdot X = c_1[A \sin kt][(2\xi - \xi^2) + 0.0281(3\xi^2 - 2\xi^3)],$$

$$\theta = T\cdot X = c_1[A \sin kt][(2\xi - \xi^2) - 0.7700(3\xi^2 - 2\xi^3)],$$

where $k^2 = \dfrac{Cm^2}{Il^2}$ and in the first solution $m^2 = 2.4680$ and in the second solution $m^2 = 23.5625$.

If more than two X functions are employed, additional natural modes of vibration can be obtained. (Compare Rayleigh's method Sec. 7, Chap. I.)

EXERCISES XXIV

1. In the application of Galerkin's method it is sometimes possible to add additional boundary conditions in addition to the necessary and sufficient conditions of the problem. These additional boundary conditions are called **secondary boundary conditions**, whereas the necessary and sufficient boundary conditions are then called **primary boundary conditions**. The aid of the secondary conditions is that they insure a more accurate solution with the choice of fewer X functions and thus reduce the labor required to solve the problem in question. To illustrate the principle let it be required to solve the illustrative example in §3·39. Obviously, there must be no contradiction between the primary and secondary conditions.

Solve, by Galerkin's method, $\frac{dy}{dx} - y = 0$ subject to the primary condition $y = 1$ for $x = 0$ and subject to the secondary condition $\frac{dy}{dx} = 1$ for $x = 0$. Then

$$Y = Y_0 + \sum_{j=1}^{5} c_j Y_j = 1 + x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5.$$

The numerical work in carrying out this solution consists in solving four linear equations in c_1, c_2, c_3, c_4. This is the same work involved in the solution of the illustrative example, but in the present problem greater accuracy is possible since a term in x^5 is available in the approximate solution.

2. Solve, by Galerkin's method, $\frac{dy}{dx} - y = 0$ subject to the conditions $y(0) = 0$ and $\frac{dy}{dx} = 1$ for $x = 0$.

3. Solve, by Galerkin's method, the uniform cantilever problem of §3·40 employing three functions X_1, X_2, and X_3.

4. The partial differential equation governing the flexural oscillations of a uniform cantilever is

$$E\,I\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = 0,$$

where E = Young's modulus,
I = moment of inertia of normal cross-section,
m = mass per unit length,
y = lateral displacement of point whose distance from the end of the beam is x,
x = distance from the fixed end, which is taken as the origin of coordinates,
t = time.

Take the boundary conditions to be

$$y = \frac{dy}{dx} = 0 \quad \text{for } x = 0, \quad \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0 \quad \text{for } x = l.$$

By Galerkin's method, obtain the periods of the two lowest modes of vibration. *Hint:* Set $x = \xi l$ and take for the appropriate functions

$$Y_r = \tfrac{1}{6}(r+2)(r+3)\,\xi^{r+1} - \tfrac{1}{3}r(r+3)\,\xi^{r+2} + \tfrac{1}{6}r(r+1)\,\xi^{r+3} \quad (r = 1, 2).$$

3·41. Galerkin's Method Extended to Non-linear Problems. The method of Galerkin is extensible to at least simple non-linear problems. Let it be required to solve $\frac{dy}{dx} = ry^2$ subject to the initial condition $y(0) = 1$. Suppose $r \leqq 1/2$ and let the interval of the solution be $0 \leqq x \leqq 1$.

If $Y_0 = 1$, $Y_1 = x$, and $Y_2 = x^2$ then

$$Y = Y_0 + \sum_{j=1}^{2} c_j\, Y_j = 1 + c_1x + c_2x^2.$$

The result of substituting $y = Y$ in the differential equation is ϵ. The conditions

$$\frac{\partial}{\partial c_i} \int_0^1 \epsilon^2 \, dx = 0, \quad (i = 1, 2)$$

yield

$$\int_0^1 [(c_1+2c_2x)-r(1+c_1x+c_2x^2)^2][1-2r(1+c_1x+c_2x^2)x]dx=0,$$

$$\int_0^1 [(c_1+2c_2x)-r(1+c_1x+c_2x^2)^2][2x-2r(1+c_1x+c_2x^2)x^2]dx=0.$$

When the indicated integrations are performed the last equations are evidently of the form of Eqs. (47) with x_1 and x_2 replaced by c_1 and c_2. They may be solved for c_1 and c_2 as a power series in r by the method of §3·12.

EXERCISE XXV

1. Carry out in detail the solution indicated in §3·41.

(9)

Method of Lalesco's Non-linear Integral Equations

The solution of non-linear problems by means of non-linear differential equations requires a knowledge of the theory of linear differential equations. On the other hand the use of the non-linear integral equations of Lalesco does not require any knowledge of linear integral equations.

3·42. Lalesco's Equation. Lalesco's non-linear integral equation is

$$\varphi(x) = f(x) + \int_0^x K[x, \xi; \varphi(\xi)]d\xi, \tag{142}$$

where $\varphi(x)$ is the unknown function or solution which is to be found and $f(x)$ and $K(x, \xi;\varphi)$ are explicit known functions of their arguments. The functions and quantities involved are subject to the following restrictions:

(*a*) The variables x and ξ and the function φ are real.

(*b*) $K[x, \xi;\varphi(\xi)]$ is a function of the real variables x, ξ, and the unknown real function φ.

(*c*) $|K(x, \xi;\varphi)| < M$ [143]
and $|K[x, \xi;\varphi_i] - K[x, \xi;\varphi_j]| < N|\varphi_i - \varphi_j|$ for $0 < \xi < x < a$ and $A - b < \varphi < A + b$, where M, N, a, and b are positive constants.

It can be shown[27] that the limit of the infinite sequence of functions

$$
\begin{aligned}
\varphi_0(x) &= f(x), \\
\varphi_1(x) &= f(x) + \int_0^x K[x, \xi; f(\xi)]d\xi, \\
\varphi_2(x) &= f(x) + \int_0^x K[x, \xi; \varphi_1(\xi)]d\xi, \\
\varphi_3(x) &= f(x) + \int_0^x K[x, \xi; \varphi_2(\xi)]d\xi, \\
&\cdots\cdots\cdots\cdots \\
\varphi_n(x) &= f(x) + \int_0^x K[x, \xi; \varphi_{n-1}(\xi)]d\xi, \\
&\cdots\cdots\cdots\cdots
\end{aligned}
\tag{144}
$$

is the solution of Eq. (142).

EXAMPLE 1. The equation

$$\frac{di}{dt} + r(i + b\,i^3) = E \cos at$$

is the equation of an inductive series circuit possessing a non-linear resistance. Let the initial condition be $i(0) = 0$.

Integrating, with respect to t, both sides of the differential equation we have

$$i = \frac{E}{a} \sin at - r\int_0^t (i + b\,i^3)dt,$$

which is an integral equation of the form of Eq. (142). Consequently, there exists a solution given by the limit of the sequence defined by Eqs. (144). Identifying the quantities of the illustrative example with those of Eqs. (144) we have

$$
\begin{aligned}
\varphi_0 &= \frac{E}{a} \sin at, \quad K = -\,r(i + b\,i^3), \\
\varphi_1 &= \frac{E}{a} \sin at - r\int_0^t \left[\frac{E}{a} \sin at + b\left(\frac{E}{a}\right)^3 \sin^3 at\right] dt, \\
&= \frac{E}{a} \sin at - \frac{rE}{a^2}(1 - \cos at) - \frac{rb}{4}\left(\frac{E}{a}\right)^3 \\
&\qquad \left[\frac{3}{a}(1 - \cos at) - \frac{r}{3a}(1 - \cos 3at)\right], \\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

[27] V. Volterra, *Leçons sur les Equations Intégrales*, p. 90; E. Picard, *Traité d'Analyse*, II, p. 340.

If b is small in the problem in question φ_1 above is an approximate solution for a finite interval of time t.

It should be noted that, by conditions (143), the interval of the independent variable t (or x) is finite. It should be pointed out that it is the *limit* of the sequence that is the solution. Hence φ_n for n sufficiently large is an approximation to the solution desired. If b is very small in the present problem then φ_1 above is an approximate solution.

EXAMPLE 2. A different approach, which is extensible to other circuit problems, is possible in example 1. Write the differential equation in the form

$$\frac{di}{dt} + ri = E \cos at - rbi^3.$$

The indicial admittance $A(t)$ for the linear circuit whose differential equation is $(p + r)i = 1$ is

$$A(t) = \frac{1}{r}(1 - e^{-rt}).$$

By Duhamel's superposition theorem the total current is

$$i = \frac{E}{r}(1 - e^{-rt}) - \frac{1}{r}\int_0^t (1 - e^{-r(t-\lambda)})\left(aE \sin a\lambda + 3rbi^2 \left.\frac{di}{dt}\right|_{t=\lambda}\right) d\lambda.$$

This equation is of the form of Eq. (142). Identifying the quantities involved with those of Eqs. (144), we have

$$\varphi_0 = \frac{E}{r}(1 - e^{-rt}), \quad K = -\frac{1}{r}(1 - e^{-r(t-\lambda)})\left(aE \sin a\lambda + 3rbi^2 \left.\frac{di}{dt}\right|_{t=\lambda}\right),$$

$$\varphi_1 = \frac{E}{r}(1 - e^{-rt}) - \frac{aE}{r}\int_0^t \sin a\lambda\, d\lambda + \frac{aE}{r} e^{-rt}\int_0^t e^{r\lambda} \sin a\lambda\, d\lambda -$$

$$3b\left(\frac{E}{r}\right)^3 \int_0^t (1 - e^{-r\lambda})^2 e^{-r\lambda}\, d\lambda + 3b\left(\frac{E}{r}\right)^3 e^{-rt}\int_0^t (1 - e^{-r\lambda})^2\, d\lambda$$

$$= \frac{E}{a^2 + r^2}(r \cos at + a \sin at - re^{-rt})$$

$$- \frac{b}{r}\left(\frac{E}{r}\right)^3 [1 + 3(\tfrac{1}{2} - rt)e^{-rt} - 3e^{-2rt} + \tfrac{1}{2}e^{-3rt}].$$

. . . .

Obviously, the values for φ_1 as given in examples 1 and 2 are different. Neither is a solution of the differential equation since only the limit of the sequence is the solution. Both φ_1 are merely approxima-

tions. The weakness of the method is due to the rapidly increasing complexity of the φ_i as i increases. In certain problems the convergence of the sequence to a very approximate solution is greatly hastened by the following fact. It can be shown that the sequence (144) will converge in the limit to the same value if $\varphi_0(x) = f(x)$ in the problem is replaced by a function which is known from physical considerations to resemble the final solution of the problem. (See §3·15, this chapter.)

EXERCISE XXVI

1. Solve, by the method of this article, the differential equation $\frac{d^2x}{d\theta^2} + r\,(x + bx^3) = E \cos\theta$, subject to the initial conditions $x(0) = C$, $x'(0) = 0$.

3·43. Systems of Non-linear Integral Equations. If a problem is reducible to the system of non-linear integral equations

$$
\begin{aligned}
u_1(x) &= \varphi_1(x) + \int_0^x K_1[x, \xi; u_1(\xi), \cdots, u_n(\xi)]d\xi,\\
u_2(x) &= \varphi_2(x) + \int_0^x K_2[x, \xi; u_1(\xi), \cdots, u_n(\xi)]d\xi, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u_n(x) &= \varphi_n(x) + \int_0^x K_n[x, \xi; u_1(\xi), \cdots, u_n(\xi)]d\xi,
\end{aligned}
\tag{145}
$$

then there exists a solution which is the limit of the infinite sequences

$$
\begin{aligned}
u_1^{(0)}(x) &= \varphi_1(x),\\
u_1^{(1)}(x) &= \varphi_1(x) + \int_0^x K_1[x, \xi; \varphi_1(\xi), \cdots, \varphi_n(\xi)]d\xi,\\
u_1^{(2)}(x) &= \varphi_1(x) + \int_0^x K_1[x, \xi; u_1^{(1)}(\xi), \cdots, u_n^{(1)}(\xi)]d\xi,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u_2^{(0)}(x) &= \varphi_2(x),\\
u_2^{(1)}(x) &= \varphi_2(x) + \int_0^x K_2[x, \xi; \varphi_1(\xi), \cdots, \varphi_n(\xi)]d\xi,\\
u_2^{(2)}(x) &= \varphi_2(x) + \int_0^x K_2[x, \xi; u_1^{(1)}(\xi), \cdots, u_n^{(1)}(\xi)]d\xi,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u_n^{(0)}(x) &= \varphi_n(x),
\end{aligned}
\tag{146}
$$

$$u_n^{(1)}(x) = \varphi_n(x) + \int_0^x K_n[x, \xi; \varphi_1(\xi), \cdots, \varphi_n(\xi)]d\xi,$$

$$u_n^{(2)}(x) = \varphi_n(x) + \int_0^x K_n[x, \xi; u_1^{(1)}(\xi), \cdots, u_n^{(1)}(\xi)]d\xi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In general, the carrying out of the sequences (146) leads to too great a complexity. However, if sufficient physical insight into a problem has been attained by means of engineering principles, oscillograms, speed curves, etc., then devices may be employed which insure sufficiently rapid convergence of the sequences (146) that two or three members of each sequence are ample for the accuracy required. Frequently, mathematical expressions of the dependent variables as functions of independent variables are not required in an engineering problem, but instead only an upper or lower limit to the range of certain quantities must be known. The present method is of value in such cases. To illustrate the principles and facts set forth in this paragraph we shall apply Eqs. (145) to the problem of dynamic braking of a synchronous machine.

EXAMPLE. The differential equations of dynamic braking (see problem 2, set III, this chapter)

$$\frac{dI}{dt} = \frac{(E - IR)[(rs_0/s)^2 + x_d x_q]}{L[(rs_0/s)^2 + x_{d'}x_q]} - \frac{2KPr^3 x_q s_0^2(x_d - x_{d'})[x_q^2 + (rs_0/s)^2]I^3}{JI_0^2 s^4[(rs_0/s)^2 + x_{d'}x_q][(rs_0/s)^2 + x_d x_q]^3}, \tag{147}$$

$$\frac{ds}{dt} = -\frac{KPrI^2}{JsI_0^2}\left[\frac{x_q^2 + (rs_0/s)^2}{[x_d x_q + (rs_0/s)^2]^2}\right], \tag{148}$$

under change of dependent variables $s = s_0 e^{-z}$, $I = \dfrac{E}{R} + I_1 e^{-y}$ (suggested by an oscillogram of the field current and by a speed curve) are reducible to the non-linear integral equations

$$y = \frac{R}{L}t + \int_0^t \frac{\mu R\,dt}{L[A_0^2 + (re^z)^2]} + \frac{2KPr^3\mu}{Js_0^2 I_1 I_0^2}\int_0^t \frac{[x_q^2 + (re^z)^2]\left[\dfrac{E}{R} + I_1 e^{-y}\right]^3 e^y\,dt}{[A^2 + (re^z)^2]^3[A_0^2 + (re^z)^2]e^{-4z}}, \tag{149}$$

$$z = \frac{KPr}{JI_0^2 s_0^2}\int_0^t \frac{\left[\dfrac{E}{R} + I_1 e^{-y}\right]^2[x_q^2 + (re^z)^2]}{e^{-2z}[A^2 + (re^z)^2]^2}\,dt \tag{150}$$

The integral equations (149) and (150) are of the form of Eqs. (145).

The problem in dynamic braking is to find an accurate expression for the number of revolutions before the rotor, running at full speed when the braking is applied, comes to rest. It will suffice if we find an upper limit to the number of revolutions provided this upper limit is within a few per cent by test of the actual stopping time. This upper limit is obtained with little labor by the method of this section although the complete integration of the differential equations by the method of §3·5 is indeed laborious. By this reduced method we shall also illustrate the principles stated in the paragraph immediately preceding the illustrative example.

It is necessary in that which follows to keep in mind that it is known from an oscillogram of the field current and from the speed curve of that rotor that both I and s are decreasing functions of the time. Consequently, both y and z are increasing functions of the time.

Identifying the notation of Eqs. (149–150) with Eqs. (146) we have

$$u_1(x) = \varphi_1(x) = \frac{R}{L}t,$$

$$u_2(x) = \varphi_2(x) = 0.$$

Write Eq. (150)

$$z = 0 + \int_0^t \frac{KPr}{JI_0^2s_0^2}\left[\frac{E}{R} + I_1e^{-y}\right]^2\left[1 - \frac{x_q(x_d - x_q)}{A^2 + (re^z)}\right]\frac{e^{2z}\,dt}{A^2 + (re^z)^2}. \quad [151]$$

The value of z satisfying

$$z = 0 + \int_0^t \frac{KPr}{JI_0^2s_0^2}\left[\frac{E}{R} + I_1e^{-y}\right]^2\left[1 - \left(\frac{A^2 - x_q^2}{A^2 + r^2}\right)\right]\frac{dt}{A^2 + r^2} \quad [152]$$

is smaller than the value of z satisfying (151) because the two values in question are identical at $t = 0$ and z from (152) is smaller than the exact value from (151) for all values of t greater than zero. This is satisfactory since an upper limit of the solution is desired. The substitution of $u_1(x) = \dfrac{Rt}{L}$ for y in (152) yields

$$z = \frac{KPr(x_q^2 + r^2)}{JI_0^2s_0^2(A^2 + r^2)^2}\int_0^t \left[\frac{E}{R} + I_1e^{-Rt/L}\right]^2 dt,$$

$$= \frac{KPr(x_q^2 + r^2)}{JI_0^2s_0^2(A^2 + r^2)^2}\left[\left(\frac{E}{R}\right)^2 t - \frac{2LI_1E}{R^2}(e^{-Rt/L} - 1) - \frac{LI_1^2}{2R}(e^{-2Rt/L} - 1)\right]. \quad [153]$$

For R/L large this value of z is approximately $z = A_1 t$ where

$$A_1 = \frac{KPr(x_q^2 + r^2)}{JI_0^2 s_0^2 (A_0^2 + r^2)^2} \cdot \left(\frac{E}{R}\right)^2.$$

Preparatory for a similar treatment of Eq. (149) it is noted that

$$\begin{aligned} (A_0^2 + r^2)\, e^{2z} &\geqq [A_0^2 + (re^z)^2], \\ (A^2 + r^2)\, e^{2z} &\geqq [A^2 + (re^z)^2], \\ (A^2 + r^2)^3 e^{6z} &\geqq [A^2 + (re^z)^2]^3 \end{aligned} \qquad [154]$$

for all $z > 0$.

If the values $z = A_1 t$ and those from Eqs. (154) are substituted in Eq. (149) an approximate value for y is obtained which is less than the true value in the differential equation. This approximate value for y is

$$\begin{aligned} y &= \frac{R}{L} t + \frac{R\mu}{L(A_0^2 + r^2)} \int_0^t e^{-2A_1 t}\, dt \\ &\quad + \frac{2KPr^3E^3}{I_1 J I_0^2 s_0^2 R^3 (A_0^2 + r^2)(A^2 + r^2)^3} \int_0^t \left[x_q^2 e^{-\left(4A_1 - \frac{R}{L}\right)t} + r^2 e^{-\left(2A_1 - \frac{R}{L}\right)t}\right] dt \\ &= \frac{R}{L} t + \alpha_1 (1 - e^{-2A_1 t}) + \alpha_2 \left(1 - e^{-\left(4A_1 - \frac{R}{L}\right)t}\right) + \alpha_3 \left(1 - e^{-\left(2A_1 - \frac{R}{L}\right)t}\right), \end{aligned} \qquad [155]$$

where

$$\alpha_1 = \frac{R\mu}{2A_1 L(A_0^2 + r^2)},$$

$$\alpha_2 = \frac{2KPr^3E^3x_q^2}{JI_1^2 s^2 I_0\, R^3 (A_0^2 + r^2)(A^2 + r^2)^3 \left(4A_1 - \frac{R}{L}\right)},$$

$$\alpha_3 = \frac{2KPr^3E^3r^2}{JI_0^2 s_0^2 I_1 R^3 (A_0^2 + r^2)(A^2 + r^2)^3 \left(2A_1 - \frac{R}{L}\right)}.$$

The integral in Eq. (150) is rewritten as

$$\int_0^z \frac{e^{-2z}[A^2 + (re^z)^2]^2}{x_q^2 + (re^z)^2}\, dz = \frac{KPr}{JI_0^2 s_0^2} \int_0^t \left[\frac{E}{R} + I_1 e^{-y}\right]^2 dt. \qquad [156]$$

The integral in Eq. (156) is

$$r^2z + \frac{x_d^2}{2}(1 - e^{-2z}) - A_2 \log [1 - A_3(1 - e^{-2z})]$$
$$= \frac{KPr}{JI_0^2s_0^2}\int_0^t \left[\frac{E}{R} + I_1e^{-v}\right]^2 dt, \quad [157]$$

where

$$A_2 = \frac{r^2}{2x_q^2}(x_d^2 + x_q^2 - 2A^2), \quad A_3 = \frac{x_q^2}{r^2 + x_q^2}.$$

The integral of the right member of Eq. (157) is evaluated by numerical integration for any particular machine giving z as a function of t. The quantity y is determined from Eq. (155) for use in (157).

The answer is

$$N = \text{number of revolutions} = \frac{s_0}{2\pi}\int_0^\infty e^{-z}\,dt.$$

The integration in the last equation is performed numerically. The test results indicate that the value of N is sufficiently accurate.

In the above example all parameters are carried nearly to the end of the solution. Thus the expressions carry information for improvement of design.

The process just completed is frequently representative of a certain type of engineering solutions. The solution of a system defining an engineering problem may fail because (*a*) its mathematical processes cannot be carried out, (*b*) if carried out they may be so complicated for computational purposes as to be practically worthless. The above method may, as in the present problem, furnish a simple answer sufficiently accurate and greatly superior to a completely accurate and complicated solution.

EXERCISE XXVII

1. Rework the illustrative example of §3·43 obtaining greater accuracy by choosing less liberal inequalities than those employed in Eqs. (154).

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Solutions by the Differential Analyzer

Only very fragmentary ideas of the nature of the differential analyzer and its solutions can be given in a page. References to the literature are given in §3·44.

3·44. Differential Analyzer. The relations expressed between the independent and dependent variables in a system of ordinary differential equations may be viewed as merely constraints imposed upon the behavior of the variables. A machine possessing parts whose motions or electrical variations represent the behavior of the variables and whose interconnections (mechanical or electrical) represent mathematical operations and relations is a differential analyzer.

The invention and design of such a machine calls forth the highest ingenuity and inventive skill.

One of the very first, if not the first, differential analyzer was invented and built at the Massachusetts Institute of Technology by V. Bush and H. Hazen. Since 1927 new and ever improved machines have been continually under development by the staff [28] of the Institute and others.[29] The latest differential analyzer of the Massachusetts Institute of Technology is nearing completion. This machine will integrate eighteenth order systems of differential equations. The integrator units are mechanical but most of the interconnections are electrical. Even the system of differential equations and their initial conditions are impressed electrically upon the machine. Although earlier machines were approximately one hundred times more rapid than analytical processes the new machine is still much more rapid. Complete descriptions of the new machine will appear presently in the literature.[30]

3·45. Solutions. A differential analyzer solution of a system of ordinary differential equations is a graph of the solution. The graph (or graphs) may actually be drawn by the machine or it may print a table of values from which the graphs may be drawn. Before the equation or system can be set up on the machine all parameters (letters) are replaced by numerical quantities. Initial conditions are introduced by the initial settings of the entities which represent the variables.

[28] V. Bush, F. D. Gage, and H. R. Stewart, "A Continuous Integraph," *J. Franklin Institute*, **203**, 63 (1927); V. Bush and H. L. Hazen, "Integraph Solution of Differential Equations," *J. Franklin Institute*, **204**, 575 (1927); K. E. Gould, "A New Machine for Integrating a Functional Product," *J. Math. Phys.*, **17**, 305 (1929); H. L. Hazen, O. R. Schurig, and M. F. Gardner, "The Massachusetts Institute of Technology Network Analyzer Design and Application to Power System Problems" (not the differential analyzer), *Trans. A.I.E.E.*, **49**, 872 (1930); V. Bush, "The Differential Analyzer. A New Machine for Solving Differential Equations," *J. Franklin Institute*, **212**, 447 (1931); T. S. Gray, "A Photo-Electric Integraph," *J. Franklin Institute*, **212**, 77 (1931).

[29] D. R. Hartree, "Differential Analyzer," *Nature*, **135**, 940 (1935); I. Travis, "Differential Analyzer Eliminates Brain Fag," *Machine Design*, **7**, 15 (July, 1935).

[30] S. H. Caldwell and Staff, forthcoming articles in *J. Applied Physics* and *J. Franklin Institute.*

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Additional Methods and References

Descriptions of additional methods and a list of references follow.

3·46. Systems of Differential Equations with Periodic Coefficients. Systems of ordinary linear differential equations with periodic coefficients frequently arise in engineering problems. An example is the system of differential equations of the armature and field currents of a synchronous machine under short circuit when all resistances are taken into account. Reference to an approximate solution is given in §3·3. Other examples are the equation of Ex. 4, problem set XII and the equation of problem 2, set VII. The last two equations have analogues in electrical engineering. Often such equations are solvable by the methods of Sec. 1. However, this is not always the case. Equations with periodic coefficients have long been of astronomical importance and consequently a large body of theory has been developed for the integration of such systems. See Ref. 14 of this article.

3·47. Non-linearity in Continuous Systems. Non-linear problems in continuous fields usually lead to non-linear partial differential equations. The methods of Poritsky and Ritz attack such problems. See Ref. 13.

3·48. References. The titles of papers are not always given in the following list when the topic heading amply identifies the subject matter.

1. Systems of Differential Equations Solved as Power Series in Parameters. F. F. Tisserand, *Mécanique Céleste*, Vol. III, Chap. 6, Gauthier-Villars et fils, Paris, 1889. E. Picard, *Traité d'Analyse*, Vol. II, pp. 255–260, Gauthier-Villars, Paris, 1883. F. R. Moulton, *Introduction to Celestial Mechanics*, pp. 264–265, Macmillan Company, New York, 1923.

2. Variation of Parameters. J. Lagrange, *Nouv. Mem. Acad. Berlin*, **5** (1774), **6** (1775), p. 190. John Bernoulli, *Acta Erud.* (1697), p. 113.

3. Differential Variations. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I, Chap. 4, E. Flammarion, Paris, 1908. F. R. Moulton, *New Methods in Exterior Ballistics*, Chap. IV, University of Chicago Press, 1926.

4. Hyperelliptic Functions. F. R. Moulton, *Am. J. Math.*, **34**, pp. 177–202. The hyperelliptic functions in this publication are in a form suitable for applications in engineering.

5. Method of Successive Approximations. E. Picard, *Traité d'Analyse*, Vol. II, p. 340, Gauthier-Villars (1905). E. Picard, *Journal de Mathématiques* [4], **6**, 197–210 (1890).

6. Series Solutions in Independent Variables of Non-linear Equations. E. T. Whittaker, *A Treatise on Analytical Dynamics*, Chap. XVI, Cambridge University Press, 1927. W. O. Pennell, *J. Math. Phys.*, **7**, 24 (1927). The method given by Pennell is an operational one and is similar to the methods of E. J. Berg which are to appear presently in book form.

7. Method of Collocation. R. A. Frazer, W. B. Jones, and S. W. Skan, R. and M. No. 1799 (2913), *A.R.C. Technical Report* (1937), Air Ministry, London: His Majesty's Stationary Office.

8. Galerkin's Method. W. J. Duncan, R. and M. 1798 (3287), *A.R.C. Technical Report* (1937), Air Ministry, London: His Majesty's Stationary Office.

9. Operational Method. "An Operational Treatment of Nonlinear Dynamical Systems," L. A. Pipes, *Journal of the Acoustical Society of America*, **10**, 29 (1938).

10. Existence Theorems. E. L. Ince, *Ordinary Differential Equations*, Chap. XIII, Longmans, Green and Co., London, 1927.

11. Non-linear Integral Equations. M. Lalesco, in the text, *Leçons sur les Equations Intégrales* by V. Volterra, Gauthier-Villars, Paris, 1913. E. Cotton, "Quasi-non-linear Differential Equations," *Bull. Soc. Math. Fr.*, **38**, 144 (1910). H. Galajikian, *Amer. Math. Soc. Bull.* **19**, 342 (1913); also *Ann. of Math.*, **2**, 16 (1915). E. Schmidt, *Math. Ann.*, **65**, 370 (1908). Lewi Tonks and I. Langmuir, "General Theory of the Plasma of an Arc," *Physical Review*, **6** (1929).

12. Mechanics. S. J. Mikina and J. P. Den Hartog, "Forced Vibrations with Non-linear Spring Constants," *Trans. A.S.M.E.*, **54**, *A.P.M.* 157 (1932). E. Trefftz, "Stability of Non-linear Systems," *Math. Ann.*, **95**, 307 (1925). J. G. Baker, "Subharmonic Resonance," *Trans. A.S.M.E.*, **54**, 162 (1932).

13. Non-linearity in Continuous Systems. Th. von Kármán, "The Engineer Grapples with Non-linear Problems," *Bulletin Am. Math. Soc.*, **46** (1940). Hillel Poritsky, "The Reduction of the Solution of Certain Partial Differential Equations to Ordinary Differential Equations," *Proceedings of the Fifth International Congress of Applied Mechanics*, John Wiley and Sons, 1939. W. Ritz, "Uber eine neue Methode zur Losung gewisser Variationsprobleme de mathematischen Physik," *Journ. f. reine u. angew. Mathematik*, **135**, 1 (1908). Also "Theorie der Transversalschwingen einer quadratischen Platte mit freien Rander," *Ann. Physik*, **28**, 737 (1909). A. F. Stevenson, "On the Theoretical Determination of Earth Resistance from Surface Potential Measurement," *Physics*, **5** (1934).

14. Linear Equations with Periodic Coefficients. F. R. Moulton, *Periodic Orbits*, Publication 161, Carnegie Institution of Washington. F. R. Moulton, *Differential Equations*, Macmillan Co., 1930. For infinite determinants see H. Poincaré, *Bulletin de la Société de France*, **14**, 77. Also E. T. Whittaker and G. W. Watson, *A Course in Modern Analysis*, Cambridge University Press.

15. Non-linear Circuits. B. van der Pol, "On Relaxation Oscillations," *Phil. Mag.*, **2**, 978 (1926). B. van der Pol, "Frequency Demultiplication," *Nature*, September, 1926. B. van der Pol, *Phil. Mag.*, **3**, 65 (1927). A. Boyajian, *General Electric Review*, **34** (1931). O. Martienssen, *Phys. Zeitschr.*, **11**, 448 (1910). P. H. Odessey and E. Weber, "Critical Conditions in Ferroresonance," *Trans. A.I.E.E.*, **57**, 444 (1938). J. R. Carson, "Theory and Calculation of Variable Electrical Systems," *Physical Review*, **17** (1921). E. G. Keller, "Resonance Theory of Series Non-Linear Control Circuits," *J. Franklin Institute*, **225**, 561. Also "Beat Theory of Non-Linear Circuits," *J. Franklin Institute*, **228**, 319.

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17. Discrete Systems. E. G. Keller, "Analytical Methods of Solving Discrete Non-Linear Problems in Electrical Engineering," *Trans. A.I.E.E.*, **60** (1941). (Contains bibliography of a hundred entries.)

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